VIBRATION ANALYSIS OF EULER AND TIMOSHENKO BEAMS USING DIFFERENTIAL TRANSFORMATION METHOD

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ABSTRACT

In this paper, a relatively new approach called the Differential Transformation Method (DTM) is applied for free vibration analysis of Euler beams and Timoshenko beams with uniform cross section. Natural frequencies are calculated for different cases of boundary conditions. MATLAB code has been developed to solve the differential equation of the beam using the Differential Transformation Method. It is found that the natural frequencies for all the boundary conditions are in excellent agreement with available solutions.

Keywords: DTM, Natural frequency, Taylor series, Vibration analysis, Euler-Timoshenko beams

I. INTRODUCTION

Many problems in science and engineering fields can be described by partial differential equations. The solution to these problems can be achieved by implementing any of the numerical and analytical methods available. The classical Taylor series method is one of the earliest analytic techniques to many problems, especially ordinary differential equations. However, for higher order derivatives, it requires a lot of symbolic calculation for the derivatives of functions and hence it takes a lot of time for computation. Here, an updated version of the Taylor series method is introduced which is called the differential transform method (DTM).

For problems of complex nature, the exact solution cannot be obtained or is hard to obtain. In such cases, approximation method is resorted. But by applying differential transformation method, even for the complex problems, the exact solution can be obtained. This method is capable of modeling any beam whose cross-sectional area and moment of inertia vary along beam with any two arbitrary functions and any type of cross-section with just one or few elements. Hence DTM can be effectively used in most of engineering applications.

DTM was first used by Zhou [1] to solve both linear and non-linear initial value problems in electric circuit analysis. C.K Chen [2] solved eigen value problems by using DTM. Fatma Ayaz[3] obtained numerical solution of linear differential equations by using DTM. Chen and Ho[4] solved eigenvalue problems for the free and transverse vibration problems of a rotating twisted Timoshenko beam under axial loading by using DTM. DTM was applied to solve a second order non-linear differential equation that describes the under damped and over damped motion of a system subject to external excitation by Jang and Chen [5]. Chen and Liu[6] considered the first order linear and non-linear two-point boundary value problems by using DTM. In the other study, Jang et al. [7] investigated the linear and non-linear initial value problems by using DTM. Hassan [8],[9] studied the solution of Sturm–Liouville eigenvalue problem and solved partial differential equations by using DTM. Bert and Zeng [10] used DTM to investigate the analysis of axial vibration of compound bars. Kurnaz et al.[11] studied n-dimensional DTM to solve the partial differential equations.

In this paper, the vibration analysis of uniform Euler-Bernoulli beams and Timoshenko Beams has been done by using DTM and the natural frequencies for various boundary conditions have been found.

II. DIFFERENTIAL TRANSFORMATION METHOD

The basic theory of Differential Transformation is stated in brief in this section.

Differential Transformation of function y(x) is defined as follows.

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \quad (2.1)$$

In (2.1), y(x) is the original function and Y(k) is the transformed function, which is called the T-function in brief. Differential inverse transformation of Y(k) is defined as follows:

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) \quad (2.2)$$

In fact, from (2.1) and (2.2), we obtain,

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \quad (2.3)$$

Equation (2.3) implies that the concept of differential transformation is derived from the Taylor series expansion. In this study we use lower-case letters to represent the original functions and upper-case letters to stand for the transformed functions (T-functions). From the definitions of (2.1) and (2.2), it is easy to prove that the transformed functions comply with the following basic mathematics operations.

Table 1: Some Basic Mathematical Operations of DTM

Original Function	*Transformed Function*	
$w(x) = y(x) \mp z(x)$	$W(k) = Y(k) \mp Z(k)$	(2.4)
$z(x) = \lambda y(x)$	$Z(k) = \lambda Y(k)$	(2.5)
$w(x) = y(x)\, z(x)$	$W(k) = \sum_{l=0}^{k} Y(l) Z(k-l)$	(2.6)
$z(x) = \frac{dy(x)}{dx}$	$Z(k) = (k+1)Y(k+1)$	(2.7)
$w(x) = \frac{d^m y(x)}{dx^m}$	$W(k) = (k+1)(k+2)..(k+m)Y(k+m)$	(2.8)
$w(x) = x^m$	$W(k) = \delta(k-m) = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$	(2.9)

In real applications, the function y (x) is expressed by a finite series and Eq. (2.2) can be written as

$$y(x) = \sum_{k=0}^{N} x^k Y(k) \quad (2.10)$$

Equation (2.10) implies that $\sum_{k=N+1}^{\infty} x^k Y(k)$ is negligibly small.

III. GOVERNING EQUATION FOR VIBRATION

3.1 Euler Beam

The governing differential equation of lateral vibration of uniform Euler beams is given by

$$EI\frac{\partial^4 v}{\partial x^4} + \bar{m}\frac{\partial^2 v}{\partial t^2} = 0 \quad (3.1)$$

By assuming harmonic motion, Let $v(x,t) = y(x)sinpt$ and converting x to non-dimensional coordinate varying from 0 to 1 by substituting $x = \frac{\bar{x}}{L}$ we get:

$$\frac{EI}{L^4}\frac{d^4 y}{dx^4}sinpt - \bar{m}yp^2 sinpt = 0 \quad (3.2)$$

Simplifying we get:

$$\frac{d^4 y}{dx^4} - \frac{\bar{m}p^2L^4}{EI}y = 0 \quad (3.3)$$

Let, $a = \frac{\bar{m}p^2L^4}{EI}$

Then equation reduces to:

$$\frac{d^4 y}{dx^4} = ay \quad (3.4)$$

Which is the governing equation for the vibration of Euler beam of uniform crossection.

3.2 Timoshenko Beam

Governing equation for Timoshenko beam for free vibration is given by;

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} - \rho \mathrm{I}\left(1 + \frac{E}{kG}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 y}{\partial t^4} = 0 \quad (3.5)$$

Using a non-dimensional coordinate by substituting $x = \frac{x}{L}$ and substituting

$w(x,t) = y(x)\sin\omega t$, we get the governing equation as

$$\frac{d^4 y}{dx^4} - af_1 y + af_2\frac{dy^2}{dx^2} + a^2 f_1 y = 0 \quad (3.6)$$

Where $\mathrm{a} = \omega^2$, $\mathrm{f}_1 = \frac{\rho A L^4}{EI}$, $\mathrm{f}_2 = \frac{\rho L^2}{E}\left(1 + \frac{E}{kG}\right)$, $\mathrm{f}_3 = \frac{\rho^2 A L^4}{kGE}$

IV. APPLYING DIFFERENTIAL TRANSFORMATION

For Euler Beam, using theorem (2.8), from Table1, differential transformation of

$$\frac{d^4 y}{dx^4} = (k+1)(k+2)(k+3)(k+4)Y[k+4]$$

Using theorem, the differential transformation of $ay = aY[k]$

Hence equation 3.4 becomes;

$$Y[k+4] = \frac{aY[k]}{(k+1)(k+2)(k+3)(k+4)} \quad (3.7)$$

Similiarly, for Timoshenko beam, Applying DTM, the governing equation 3.6 becomes

$$Y[k+4] = \frac{\{(af1 - a^2 f3)Y[k] - (k+1)(k+2)af2Y[k+2]\}}{(k+1)(k+2)(k+3)(k+4)} \quad (3.8)$$

V. APPLYING BOUNDARY CONDITIONS

5.1 Simply Supported At Both Ends

Deflection and curvature at both ends=0

$y(0) = 0; y''(0) = 0$ (3.9)

$y(1) = 0; y'' (1) = 0$ (3.10)

The DT equivalents of (3.9) gives

Y[0] =0

Y[2]=0

Let Y[1]=c; Y[3]=d; Then,

Y[0] = 0

Y[1] = c

Y[2] = 0

Y[3] = d

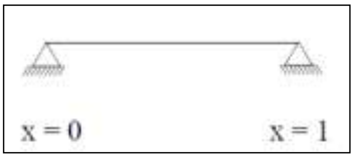

Figure 3.1 simply supported beam

Using the recursive relation of Eq.(3.7) for Euler Beam, Eq (3.8) for Timoshenko beam, we can get the remaining terms.

The DT equivalents of (3.10) gives

$$\sum_{k=0}^{\infty} Y[k] = 0 \quad (3.11)$$

$$\sum_{k=0}^{\infty} k(k-1)Y[k] = 0 \quad (3.12)$$

Both equations give nonlinear equation in terms of ‘a’ and linear in terms of ‘c’ and ‘d’.

Putting the boundary conditions (3.11) and (3.12) in matrix form, we get,

$$\begin{bmatrix} aa & bb \\ cc & dd \end{bmatrix} \begin{Bmatrix} c \\ d \end{Bmatrix} = \{0\}$$

Where *aa* and *cc* are the coefficients of c, bb and dd are the coefficients of d

Since $\begin{Bmatrix} c \\ d \end{Bmatrix} \neq 0$, the determinant of matrix $\begin{bmatrix} aa & bb \\ cc & dd \end{bmatrix}$ must be equal to zero.

Hence,

$$aa \times dd - bb \times cc = 0$$

Which gives a polynomial in ‘a’. Depending upon the number of terms N taken, we get a higher degree polynomial in ‘a’. Solving for a the frequency is obtained as $p = \frac{1}{L^2}\sqrt{\frac{EIa}{m}}$. Same procedure is applied for different boundary conditions as below.

5.2 Fixed-Fixed Supports

Slope and deflection at both ends=0

$y(0) = 0; y'(0) = 0$ (3.13)

$y(1) = 0; y' (1) = 0$ (3.14)

The DT equivalents of (3.13) gives

Y[0] =0

Y[1]=0

Figure3.2 Fixed-Fixed beam

Let Y[2]=c; Y[3]=d; Then,

Y[0] = 0

Y[1] = 0

Y[2] = c

Y[3] = d

The DT equivalents of (3.14) gives

$\sum_{k=0}^{\infty} Y[k] = 0$ (3.15)

$\sum_{k=0}^{\infty} kY[k] = 0$ (3.16)

5.3 Fixed-Roller Supports

Curvature and deflection at roller end =0

Slope and deflection at fixed end =0

y(0) = 0; y''(0) = 0 (3.17)

y(1) = 0; y' (1) = 0 (3.18)

The DT equivalents of (3.17) gives

Y[0] =0

Y[2]=0

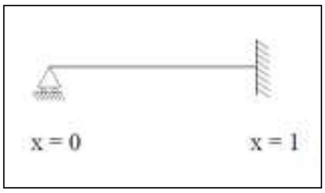

Figure3.3 Fixed-Roller beam

Let Y[1]=c; Y[3]=d; Then,

Y[0] = 0

Y[1] = c

Y[2] = 0

Y[3] = d

The DT equivalents of (3.18) gives

$\sum_{k=0}^{\infty} Y[k] = 0$ (3.19)

$\sum_{k=0}^{\infty} kY[k] = 0$ (3.20)

5.4 Fixed-Free Supports (Cantilever)

At fixed end,

Deflection and slope=0

y(0) = 0; y'(0) = 0 (3.21)

At the free end,

Bending moment= $EI\frac{\partial^2 v}{\partial x^2} = 0$

Shear force = $EI\frac{\partial^3 v}{\partial x^3} = 0$

$\therefore$ DT equivalents of Bending moment & Shear force gives Y''(1) = 0; Y''' (1) = 0 (3.22)

The DT equivalents of (3.21) gives

Y[0] =0

Y[1]=0

Figure3.4 Fixed-Free beam

Let Y[1]=c; Y[3]=d; Then,

Y[0] = 0

Y[1] = 0

Y[2] = c

Y[3] = d

The DT equivalents of (3.22) gives

$\sum_{k=0}^{\infty} k(k-1)Y[k] = 0$ (3.23)

$\sum_{k=0}^{\infty} k(k-1)(k-2)Y[k] = 0$ (3.24)

VI. NUMERICAL RESULTS

6.1 Euler Beam

Since $a = \frac{mp^2L^4}{EI}$, the frequency is obtained as $p = \frac{1}{L^2}\sqrt{\frac{EIa}{m}} = \frac{k}{L^2}\sqrt{\frac{EI}{m}}$ where $k = \sqrt{a}$

K value obtained for Euler beam is listed in Table 2.

Table2: Value of A and K Obtained For Different Boundary Conditions of an Euler Beam

Boundary conditions	'a' obtained	$k = \sqrt{a}$
Simply Supported at both ends	97.4090910	9.8696
fixed-fixed supports	500.563901	22.373
fixed-roller supports	237.721067	15.418
fixed-free supports	485.51881	22.034

6.2 Timoshenko Beam

Assuming unit width, Poisson's ratio $\nu = 0.3$ and the shear coefficient of the beam k=5/6, Modulus of Elasticity E = 200Gpa, length of the beam L =10m, mass density $\rho = 7800 kg/m3$, Moment of Inertia I =8.333e-5m^4, Area A= 0.1m^2 (h/L=0.01) we can find the value of nondimensional frequency parameter λ from the formula,

$$\lambda^2 = \omega L^2\sqrt{\frac{\rho A}{EI}}, \quad \text{where } \omega = \sqrt{a}$$

λ value obtained for Timoshenko Beam is listed in Table 3.(Shear deformation at the ends being small, Euler beam boundary conditions have been adapted.)

Table3: Value of A and λ Obtained For Different Boundary Conditions of Timoshenko Beam

Boundary conditions	'a' obtained	λ
Simply Supported at both ends	208.09	3.1417
fixed-fixed supports	1069.311	4.730
fixed-roller supports	507.85	3.926
fixed-free supports	26.418	1.875

VII. CONCLUSION

Natural frequencies of uniform Euler and Timoshenko beam for various boundary conditions are obtained using DTM. The method is successfully implemented in MATLAB for convergence. Number of terms required for convergence is generally taken as 30. It is found that the natural frequencies obtained, are in excellent agreement with closed form solutions. Comparison of k obtained by DTM and closed form solutions for Euler Beam are listed in Table 3. Also comparison of nondimensionalized frequency parameter λ obtained by DTM and closed form solution for Timoshenko beam are listed in Table 5.

Table4: Comparison Of DTM With Closed Form Solution For Euler Beam

Boundary conditions	Closed form solution	DTM method
Simply supported	9.87	9.869
Fixed- Fixed	22.4	22.373
Fixed-roller	15.4	15.418
Fixed-free	3.52	3.516

Table 5: Comparison Of DTM With Closed Form Solution For Timoshenko Beam

Boundary conditions	Closed form solution	DTM method
Simply supported	3.141	3.141
Fixed- Fixed	4.73	4.73
Fixed-roller	3.927	3.926
Fixed-free	1.875	1.875

Based on the results presented, it can be demonstrated that the Differential Transformation Method is convenient and efficient in solving the vibrations of beams with good accuracy using fewer number of terms.

It is expected that DTM will be more promising for further development into efficient and flexible numerical techniques for solving practical engineering problems in future.

VIII. ACKNOWLEDGEMENT

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