STEADY STATE SOLUTION OF ONE DIMENSIONAL HEAT EQUATION USING MARKOV CHAINS

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ABSTRACT

Although the pedagogical value of introducing numerical methods such as the finite element methods, finite difference methods, moment methods, and Markov Chain methods in potential problems. In this paper, we make use of Markov chain method to find the steady state solution of one dimensional heat equation in the rod at equal distance nodes in elementary matrix operations.

Keywords: Markov Chain, One Dimensional Steady State Equation.

I. INTRODUCTION

There are different methods to solve Steady State Solution of One Dimensional Heat Equation like the finite element methods, finite difference methods, moment method and Markov chains method [3]. We have attempted to make use of a simple technique modifying the Markov Chains method to find the Steady State solution of one dimensional heat equation. The technique basically calculates the transition probabilities using Markov chains.

II. MARKOV CHAINS

Consider a sequence of random variables $X_0, X_1, ...,$ and suppose that the set of possible values of these random variables is $\{0,1,...,M\}$. It will be helpful to interpret X_n as being the state of some system at time n, and, in accordance with this interpretation, we say that the system is in state i at time n if $X_n = i$. The sequence of random variables is said to form a Markov Chain if each time the system is in state i there is some fixed probability – call it p_{ij} – it will next be in state j. That is, for all $i_0, ..., i_{n-1}, i, j$.

$$p_{ij} = P\left(x_{n+1} = j / x_n = i, x_{n-1} = i_{n-1}, ..., x_1 = i_1, x_0 = i_0\right)$$

$$= P\left(x_{n+1} = j / x_n = i\right), j \in X, n = 0,1,2,...$$

The Markov chain is characterized by its transition probability matrix P, defined by:

$$P = \begin{pmatrix} p_{00} & p_{01} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

P is a stochastic matrix, meaning that the sum of the elements in each row is unity, i.e. $\sum_{j \in X} p_{ij} = 1, i \in X$

A Markov process is a type of random process that is characterized by the memory less property [6-9]. It is a process evolving in time that remembers only the most recent past and whose conditional probability distributions are time invariant. Markov Chains are mathematical models of this kind of process. The Markov Chain is the random walk and the states are the grid nodes. The transition probability p_{ij} is the probability that a random-walking particle at node i moves to node j.

Suppose that this method is to be applied in Steady State One dimensional heat Equation:

$$\frac{d^2U}{dx^2} = 0 \text{ in the real line X.} \qquad (1)$$

Subject to dirichlet boundary conditions:

$$U = U_P \text{ on boundary B} \quad (2)$$

The real line X is divided into n nodes. If we assume that there are f free nodes (non-absorbing) and p fixed (absorbing) nodes, the size of the transition matrix p is n. Where $n = f + p$. (3)

If the absorbing nodes are numbered first and the non-absorbing states are numbered last, the $n \times n$ transition matrix becomes:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} \qquad (4)$$

Where the $f \times p$ matrix R represents the probabilities of moving from non-absorbing nodes to absorbing ones; the $f \times f$ matrix Q represents the probabilities of moving from one non-absorbing node to another; I is the $p \times p$ identity matrix representing transitions between the absorbing nodes $(p_{ii} = 1 \& p_{ij} = 0)$; and 0 is the null matrix showing that there are no transitions from absorbing to non-absorbing nodes. For the solution of equation in (1), we obtain the elements of Q from (4) as:

$$Q = \begin{cases} \frac{1}{2}, if \quad i \quad is\, directly \quad connected \quad to \quad j \\ 0, if \quad i = j\, or \quad i \quad is\, not \quad directly \quad connected\, to \quad j \end{cases} \qquad (5)$$

The same applies to R_{ij} except that j is an absorbing node.

The probability matrix B is: $B = [R \quad Q]$ (6)

Where R_{ij} is the probability that a random-walking particle originating from a non-absorbing node i will end up at the absorbing node j. B is a $f \times n$ matrix and is stochastic like the transition probability matrix, i.e.

$$\sum_{j=1}^{n_p} B_{ij} = 1, i = 1, 2, ..., f. \qquad (7)$$

If U_f and U_n contain potentials at the free and fixed nodes respectively, then

$$U_f = BU_P \qquad (8)$$

The equation (8) will give the f number of equations where f is the number of free nodes. Solving these equations we get the solution of free nodes U_f .

III. ILLUSTRATIVE EXAMPLES

Two simple examples will corroborate the claims above. Neither requires any computer programming.

Example 1: Consider an infinitely long homogeneous rod with surface are insulated of sides A and B with $0^\circ C$ and $100^\circ C$ respectively. We wish to determine the heat potential at the centre. Mathematically, the problem is posed as: $\frac{d^2U}{dx^2} = 0$ Subject to U (0) = 0, U(x) = 100

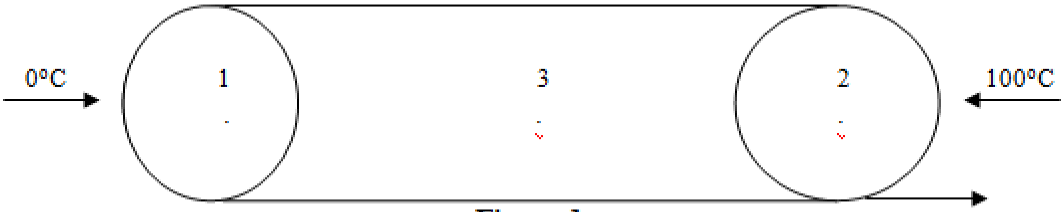

Figure 1

To apply Markov Chain Technique, we number the nodes in equal distance as in fig. 1, node 3 is the only free node so that $n_f = 1, n_p = 2$. The transition probability matrix is given by:

$$P = \begin{matrix} & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & \frac{1}{2} & \frac{1}{2} & 0 \end{matrix}$$

It is evident that:

$$Q = 0 \ and \ R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Thus: $$B = [R \quad Q] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

And $$U_f = BU_n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ u_3 \end{bmatrix}$$

$$u_3 = \tfrac{1}{2} \times 0 + \tfrac{1}{2} \times 100$$

$$u_3 = 50$$

This gives the centre node temperature which is equal to actual method value of the centre node.

Example 2: This is the same problem as in example 1 except that we are now to calculate the heat potential equal distance three points as shown in fig.2.

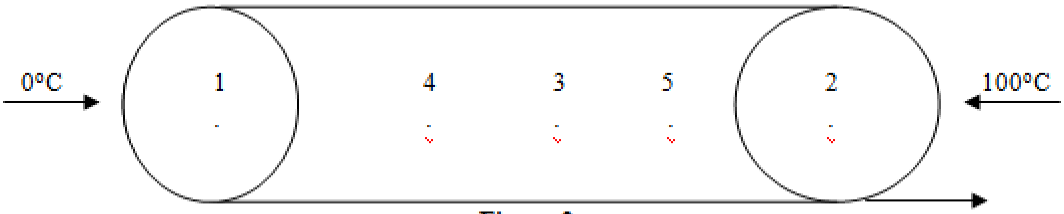

Figure 2

The transition probability matrix is obtained by inspection as:

$$P = \begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 4 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 5 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{matrix}$$

From P, we obtain:

$$R = \begin{matrix} & 1 & 2 \\ 3 & 0 & 0 \\ 4 & \frac{1}{2} & 0 \\ 5 & 0 & \frac{1}{2} \end{matrix} \quad \text{And} \quad Q = \begin{matrix} & 3 & 4 & 5 \\ 3 & 0 & \frac{1}{2} & \frac{1}{2} \\ 4 & \frac{1}{2} & 0 & 0 \\ 5 & \frac{1}{2} & 0 & 0 \end{matrix}$$

$$B = [R \quad Q] = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Now $U_f = BU_p$

$$\begin{pmatrix} u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 100 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}$$

Hence, we have $2u_3 - u_4 - u_5 = 0$

$$2u_4 - u_3 = 0$$

$$2u_5 - u_3 = 100$$

Solving the above equation by iterative method we get $u_3 = 50, u_4 = 25 \& u_5 = 75$

Table 1

Node	**Markov Chain Solution**	**Exact Solution**
3	50	50
4	25	25
5	75	75

From Table 1 it is evident that Markov Chain Solution gives the exact solution.

IV. CONCLUSION

This paper has presented a means for using Markov Chain Method to solve steady state solution of one dimensional heat equation in the rod at equal nodes. The approach uses Markov Chains to calculate the transition probabilities. This approach is not subject to randomness because a random-number generator is not required. Without knowing the length of the rod the temperature in equal distance can be found out using this method.

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