International Journal of Advance Research in Science and Engineering
Vol. No.5, Special Issue No. (01), February 2016
www.ijarse.com
IJARSE
ISSN 2319 - 8354

A FIXED POINT THEOREM FOR SELF MAPPING IN BANACH SPACE

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ABSTRACT

In this paper we consider X be a Compact subspace of a Banach space and f be a self mapping of X. We introduce condition for self mapping f such that f has a unique fixed point in X. In the other words, we established fixed point theorem with help of self mapping, satisfying contractive type of condition.

Keyword: Banach Space, Contraction Mapping, Fixed Point Theorem, Self Mapping,.

I. INTRODUCTION

In recent years, a good deal of work have been done in nonlinear analysis. The study of non contraction mapping concerning the existence of fixed points draw attention of various authors in non linear analysis. Fixed point theorem is very important in the solution of differential equations. The most famous of fixed point theorem is Brouwer's fixed point theorem. Also, a large variety of the problems of analysis and applied mathematics are used to find solutions of non-linear functional equations which can be formulated in terms of finding the fixed points of a non-linear mapping. (see [1,2,3,4]).

1.1. Definition: Let M be a vector space over the real or complex numbers (the scalars). A mapping $||\cdot||$: M → R+ is called a norm provided that the following conditions hold:

a) $||x|| = 0$, if and only if, $x = 0$ $(\in M)$

b) $||\alpha x|| = |\alpha|\ ||x||$, $\forall\, x$ $(\in M)$

c) $||x+y|| \leq ||x|| + ||y||$, $\forall\, x, y$ $(\in M)$.

If M is a vector space and $||\cdot||$ is a norm on M, then the pair (M, $||\cdot||$) is called a normed vector space. If M is a vector space and $||\cdot||$ is a norm on M, then M becomes a metric space if we define the metric d by $d(x, y) = ||x-y||$, $\forall\, x,y$ $(\in M)$. A normed vector space which is a complete metric space, with respect to the metric d defined above, is called a Banach space.

1.2 Definition: Let X be a metric space equipped with a distance d. A map f: X → X is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that $d(f(x_1), f(x_2)) \leq \lambda\, d(x_1, x_2)$, $\forall\, x_1, x_2 \in X$. The smallest λ for which the above inequality holds is the Lipschitz constant of f. If $\lambda = 1$ f is said to be non-expansive, if $\lambda < 1$ f is said to be a contraction [5,6].

First we see some fundamental results:

1.3 Theorem [**Banach's fixed point theorem**] **:** Let f be a contraction on a Banach space X, then f has a unique fixed point. In the other words, If (X, d) be a complete metric space and f: X → X a contraction, then f has a unique fixed point. [7,8].

1.4Theorem: Kannan in [9] proved that "If f is self mapping of a complete metric space X into itself satisfying: $d(f(x), f(y)) \le \alpha[\, d(f(x), x)+d(f(y), y)]$, for all $x,y \in X$ and $\alpha \in [0,1/2\,]$, then f has a unique fixed point in X".

1.5Theorem: Fisher in [9] proved the result with-
$d(f(x), f(y)) \le \alpha[\, d(f(x), x)+d(f(y), y)] + \beta d(x,y)$ for all x , $y \in X$ and $\alpha, \beta \in [0,1/2\,]$, then f has a unique fixed point in X.

1.6Theorem: A similar conclusion was also obtained by Chaterjee [10]-
$d(f(x), f(y)) \le \alpha[\, d(f(x), y)+d(f(y), x)]$,for all x , $y \in X$ and $\alpha \in [0,1/2\,]$, then f has a unique fixed point in X.

II. A FIXED POINT THEOREM FOR SELF MAPPING IN BANACH SPACE

2.1 Theorem: Let X be a complete metric space and let f be a self mapping of X into itself which satisfies the following condition: $d(f(x), f(y)) \le \alpha[\, d(f(x), y) + d(f(y), x)] - d(x,y)$
for all x, $y \in X$ and $\alpha \in [1/4,3/8\,]$, then f has a unique fixed point in X, where g is self mapping in X such that $g(x)= f(2x) - x$.

Proof:
$$
\begin{aligned}
d(g(x),g(y)) &= \| g(x)-g(y) \| \\
&= \|f(2x)-f(2y)-x+ y \| \\
&\le \|f(2x)-f(2y) \| +\| x-y \| \\
&\le d(f(2x), f(2y)) + d(x, y) \\
&\le \alpha\, [\, d(f(2x),2y)+d(f(2y), 2x)\,] - d(2x,2y) + d(x, y) \\
&\le \alpha\, [\| f(2\,x)-2y \|+\| f(2y)-2x \|\,] - \| 2x-2y \|+\| x-y \| \\
&\le \alpha\, [\| g(\,x)+ x -2y \|+\| g(y)+y-2\,x \|] - 2\| x-y \|+\| x-y \| \\
&\le \alpha\, [\| g(\,x) -x \|+\|g(y) -y \|]+(4\,\alpha - 1)\| x-y \| \\
&\le \alpha\, [\, d(g(x), x)+d(g(y), y)]+ \beta d(x,y)
\end{aligned}
$$
[Let $4\,\alpha - 1= \beta \Rightarrow 0\le\beta\le 1/2$]

$$\Rightarrow \quad d(g(x),g(y))\le \alpha[\, d(g(x), x)+d(g(y), y)]+ \beta d(x,y).$$

By theorem 1.5, "g" has a unique fixed point p in X. This means g(p) = p. Therefore, f(2p) – p = p. Hence, f(2p)=2p. Let c=2p, then f(c)=c . Therefore, f has a unique fixed point in X. The proof of theorem in this case is complete.

2.2 Theorem: Let X be a complete metric space and let f be a self mapping of X into itself which satisfies the following condition: $d(f(x), f(y)) \le \alpha[\, d(f(x),y)+d(f(y), x)] - d(x,y)$
for all x, $y \in X$ and $\alpha \in [1/6, 1/4\,]$, then f has a unique fixed point in X, where g is self mapping in X such that $g(x)= f(3x) -2x$.

Proof:
$$
\begin{aligned}
d(g(x),g(y)) &= \|g(x)-g(y)\| \\
&= \|f(3x)-f(3y)-2x+2\, y \| \\
&\le \|f(3x)-f(3y)\|+2\|x-y\| \\
&\le d(f(3x), f(3y)) + 2d(x, y)
\end{aligned}
$$

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IJARSE
ISSN 2319 - 8354

$\leq \alpha$ [d(f(3x),3y)+d(f(3y), 3x)] – $d(3x,3y)$ + 2d(x, y)

$\leq \alpha$ [||f(3 x)−3y||+||f(3y)−3x||] – ||3x−3y||+2||x−y ||

$\leq \alpha$ [||g(x)+ 2x −3y||+||g(y)+2y−3 x ||]– 3||x−y||+2||x−y||

$\leq \alpha$ [||g(x) −x ||+||g(y) −y||]+(6 α – 1)||x−y ||

$\leq \alpha$ [d(g(x), x)+d(g(y), y)]+ βd(x,y)

[Let 6 α – 1= $\beta \Rightarrow 0 \leq \beta \leq 1/2$]

$\Rightarrow$ d(g(x),g(y))≤ α[d(g(x), x)+d(g(y), y)]+ βd(x,y).

By theorem 1.5, “g” has a unique fixed point p in X. This means g(p) = p. Therefore, f(3p) − 2p=p. Hence, f(3p)=3p. Let c = 3p, then f(c)=c . Therefore, f has a unique fixed point in X. The proof of theorem in this case is complete.

2.3 Theorem: Let X be a complete metric space and let f be a self mapping of X into itself which satisfies the following condition: d(f(x), f(y)) ≤α[d(f(x),y)+d(f(y), x)] – $d(x,y)$

for all x, y ∈ X and α ∈ [1/8, 3/16], then f has a unique fixed point in X, where g is self mapping in X such that g(x)= f(4x) − 3x.

Proof: d(g(x),g(y)) = ||$g(x)$−$g(y)$||

= ||f(4x)−f(4y)−3x+3 y ||

≤ ||f(4x)−f(4y)||+3||x−y||

≤ d(f(4x), f(4y)) + 3d(x, y)

$\leq \alpha$ [d(f(4x),4y)+d(f(4y), 4x)] – $d(4x,4y)$ +3 d(x, y)

$\leq \alpha$ [||f(4 x)−4y||+||f(4y)−4x||] – ||4x−4y||+3||x−y||

$\leq \alpha$ [||g(x)+ 3x −4y||+||g(y)+3y−4 x ||]– 4||x−y||+3||x−y||

$\leq \alpha$ [||g(x) −x||+||g(y) −y||]+(8 α – 1)||x−y||

$\leq \alpha$ [d(g(x), x)+d(g(y), y)]+ βd(x,y)

[Let 8 α – 1= $\beta \Rightarrow 0 \leq \beta \leq 1/2$]

$\Rightarrow$ d(g(x),g(y))≤ α[d(g(x), x)+d(g(y), y)]+ βd(x,y).

By theorem 1.5, “g” has a unique fixed point p in X. This means g(p) = p. Therefore, f(4p) −3p=p. Hence, f(4p)=4p. Let c = 4p, then f(c)=c . Therefore, f has a unique fixed point in X. The proof of theorem in this case is complete.

2.4 Theorem: Let X be a complete metric space and let f be a self mapping of X into itself which satisfies the following condition: d(f(x), f(y)) ≤ α[d(f(x),y)+d(f(y), x)] – $d(x,y)$

for all x, y ∈ X and α ∈ [1/2k, 3/4k], then f has a unique fixed point in X, where g is self mapping in X such that g(x)= f(kx) − (k−1)x.

Proof: d(g(x),g(y)) = ||$g(x)$−$g(y)$||

= ||f(kx)−f(ky)−(k−1)x+(k−1) y ||

≤ ||f(kx)−f(ky)||+ (k−1)||x−y||

≤ d(f(kx), f(ky)) + (k−1)d(x, y)

$\leq \alpha$ [d(f(kx),ky)+d(f(ky), kx)] – d(kx,ky) +(k−1) d(x, y)

$\leq \alpha$ [||f(k x)− ky||+||f(ky)− kx||] – ||kx−ky||+(k−1)||x−y||

$\leq \alpha$ [||g(x)+ (k−1)x −ky||+||g(y)+ (k−1)y−k x ||]– k||$x-y$|| +(k−1)||$x-y$||

$\leq \alpha$ [||g(x) −x ||+||g(y) −y||]+(2k α – 1)||$x-y$||

$\leq \alpha$ [d(g(x), x)+d(g(y), y)]+ βd(x,y)

[Let (2k α – 1)= $\beta \Rightarrow 0\leq\beta\leq 1/2$]

$\Rightarrow$ d(g(x),g(y)) $\leq$ α[d(g(x), x)+d(g(y), y)]+ βd(x,y).

By theorem 1.5, "g" has a unique fixed point p in X. This means g(p) = p. Therefore, f(kp) − (k−1)p = p. Hence, f(kp) = kp. Let c = kp, then f(c)=c . Therefore, f has a unique fixed point in X. The proof of theorem in this case is complete.

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