ON SIMULTANEOUS APPROXIMATION BY BETA OPERATORS

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ABSTRACT

For last three decades applications of beta operators in the area of approximation theory is an active area of research. In the present paper, we obtain asymptotic formula for modified beta operators in linear simultaneous approximation. To establish our result, we have used the technique of linear approximating method, namely, Steklov mean.

Keywords- Simultaneous approximation, Linear combinations, Linear positive operators, Steklov mean.

MATHEMATICAL SUBJECT CLASSIFICATION: 41A25, 41A36.

I. INTRODUCTION

In approximation theory, beta operators have been studied for last three decades. Beta operators were introduced and studied by several researchers [1, 3 , 7, 8]. In the present paper we study an asymptotic formula in simultaneous approximation for the linear combinations of the operators introduced by Gupta et al. [2]. The modified beta operators introduced by Gupta et al. [2] are defined as

$$B_n(f,x)=\int_0^\infty W_n(x,t)f(t)dt \quad , \quad x\in[0,\infty) \tag{1.1}$$

where

$$W_n(x,t)=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)b_{n,v}(t) \quad , \quad b_{n,v}(t)=\frac{1}{\beta(v,n+1)}t^{v-1}(1+t)^{-n-v-1}$$

and $\beta(v,n+1)=(v-1)!n!/(n+v)!$ the Beta function.

It is easily checked that the operators defined by (1.1) are linear positive operators and it is obvious that $B_n(1,x)=1$. Also it is observed that the order of approximation by operators (1.1) is, at best O(n^{-1}), howsoever smooth the function may be. Thus, to improve the order of approximation we may consider some combinations of the operators (1.1). One approach to improve the order of approximation is the iterative combinations due to Micchelli [5], who improved the order of approximation of Bernstein polynomials. However, we cannot apply this approach to the operators (1.1) because for these operators (1.1), we not have $B_n(t-x,x)=0$, which is essential property for making iterative combinations. Yet another approach for improving the order of approximation is the technique of linear combinations which was first considered by May [4] to improve the order of approximation for exponential type operators. In the present paper, we use the later approach, which described as:

Let d_0, d_1, d_2… d_k be (k+1) arbitrary but fixed distinct positive integers. Then the linear combination $B_n(f,k,x)$ of $B_{d_j n}(f,x)$, j = 0, 1, 2…n is defined as

$$B_n(f,k,x)=\frac{1}{\Delta}\begin{vmatrix} B_{d_0 n}(f,x) & d_0^{-1} & d_0^{-2} & \ldots\ldots\ldots\ldots & d_0^{-k} \\ B_{d_1 n}(f,x) & d_1^{-1} & d_1^{-2} & \ldots\ldots\ldots\ldots & d_1^{-k} \\ \ldots\ldots\ldots\ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots\ldots\ldots\ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ B_{d_k n}(f,x) & d_k^{-1} & d_k^{-2} & \ldots\ldots\ldots\ldots & d_k^{-k} \end{vmatrix}$$

(1.2)

where $$\Delta=\begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \ldots\ldots\ldots\ldots & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & \ldots\ldots\ldots\ldots & d_1^{-k} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 1 & d_k^{-1} & d_k^{-2} & \ldots\ldots\ldots\ldots & d_k^{-k} \end{vmatrix}$$

The above expression (1.2) after simplification may be written as

$$B_n(f,k,x)=\sum_{j=0}^{k} C(j,k)\,B_{d_j n}(f,x)$$

(1.3)

where $C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i}$, $k\neq 0$ and C(0,0)=1.

Some basic properties of $b_{n,v}(x)$ are as follows

(i). $$\int_0^{\infty} t^2\, b_{n,v}(t)\;dt=\frac{v(v+1)}{n(n-1)}$$

(1.4)

(ii). $$\sum_{v=1}^{\infty} b_{n,v}(x)=(n+1)$$

(1.5)

(iii). $$\sum_{v=1}^{\infty} v\,b_{n,v}(x)=(n+1)[1+(n+2)x]$$

(1.6)

(iv). $\sum_{v=1}^{\infty} v^2 b_{n,v}(x) = (n+1)[1+3(n+2)x+(n+2)(n+3)x^2]$

(1.7)

(v). $x(1+x)b'_{n,v}(x) = [(v-1)-(n+2)x]b_{n,v}(x)$

(1.8)

where $n \in N$ and $x \in [0,\infty)$.

Throughout this paper, we may assume that $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$ and $I_i = [a_i, b_i]$ where i =1, 2, 3.

Let $H[0,\infty)$ be the class of all measurable functions defined on $[0,\infty)$ satisfying

$$\int_0^{\infty} \frac{|f(t)|}{(1+t)^{n+1}} dt < \infty \quad \text{for some positive integer n.}$$

Obviously the class $H[0,\infty)$ is bigger than the class of all lebesgue integrable functions on $[0,\infty)$. Therefore the operators (1.1) may be applicable for studying a larger class.

The main object of the present paper is to study a Voronovskaja-type asymptotic formula in simultaneous approximation for linear combinations of the operators (1.1).

II. AUXILIARY RESULTS

To prove our main results, we shall require the following preliminary results:

Lemma 2.1. For $m \in N^0$ (the set of non-negative integers) and $n > m$, let the function $\mu_{n,m}(x)$ be defined as

$$\mu_{n,m}(x) = \left(B_n(t-x)^m, x\right) = \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} b_{n,v}(t)(t-x)^m dt .$$

Then $\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \frac{2x+1}{n}$, and there holds the recurrence relation

$$(n-m)\,\mu_{n,m+1}(x) = x(1+x)[\mu^{/}_{n,m}(x) + 2m\mu_{n,m-1}(x)] + (m+1)(1+2x)\mu_{n,m}(x) .$$

Moreover, we have the following consequences about $\mu_{n,m}(x)$:

(i). $\mu_{n,m}(x)$ is a polynomial in x of degree m,

(ii). for every $x \in [0,\infty)$, $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$

where [α] denotes the integral part of α.

Consequently, on using Holder's inequality, we have from this recurrence relation that

$$B_n\left(|t-x|^r, x\right) = O\left(n^{-r/2}\right) \text{ for each r >0 and for every fixed } x \in [0,\infty) .$$

Proof. Since $\mu_{n,m}(x) = B_n\left((t-x)^m, x\right)$, therefore, using linearity property, we have

$$\mu_{n,0}(x) = B_n\left((t-x)^0, x\right) = B_n(1,x) = 1 \qquad \text{and}$$

$$\mu_{n,1}(x) = B_n\left((t-x), x\right) = B_n(t,x) - xB_n(1,x) = \frac{1+2x}{n}$$

To prove the recurrence relation we shall make the use of the following identity

$$x(1+x)b_{n,v}^{/}(x) = [(v-1)-(n+2)x]b_{n,v}(x)$$

Now, we have

$$x(1+x)\mu_{n,m}^{/}(x) = \frac{1}{(n+1)}\sum_{v=1}^{\infty} x(1+x)b_{n,v}^{/}(x)\int_0^{\infty} b_{n,v}(t)(t-x)^m\,dt$$

$$-m\frac{1}{(n+1)}\sum_{v=1}^{\infty} x(1+x)b_{n,v}(x)\int_0^{\infty} b_{n,v}(t)(t-x)^{m-1}dt$$

or

$$x(1+x)[\mu_{n,m}^{/}(x)+m\mu_{n,m-1}(x)] = \frac{1}{(n+1)}\sum_{v=1}^{\infty}[(v-1)-(n+2)x]b_{n,v}(x)\int_0^{\infty} b_{n,v}(t)(t-x)^m\,dt$$

$$= \frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty}[(v-1)-(n+2)t+(n+2)(t-x)]b_{n,v}(t)(t-x)^m\,dt$$

$$= \frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} t(1+t)b_{n,v}^{/}(t)(t-x)^m\,dt + (n+2)\mu_{n,m+1}(x)$$

$$= \frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty}[(t-x)^2+(1+2x)(t-x)+x(1+x)]b_{n,v}^{/}(t)(t-x)^m\,dt + (n+2)\mu_{n,m+1}(x)$$

$$= \frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b^{/}{}_{n,v}(t)(t-x)^{m+2}\,dt + \frac{(1+2x)}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b^{/}{}_{n,v}(t)(t-x)^{m+1}\,dt$$

$$+\frac{x(1+x)}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} b^{/}{}_{n,v}(t)(t-x)^m\,dt + (n+2)\mu_{n,m+1}(x)$$

$$= -(m+2)\mu_{n,m+1}(x) - (m+1)(1+2x)\mu_{n,m}(x) - mx(1+x)\mu_{n,m-1}(x) + (n+2)\mu_{n,m+1}(x)$$

Thus, we get the required recurrence relation

$$(n-m)\mu_{n,m+1}(x) = (m+1)(1+2x)\mu_{n,m}(x) + x(1+x)[\mu_{n,m}^{/}(x) + 2m\mu_{n,m-1}(x)]$$

The other consequences follow easily from the above recurrence relation.

Lemma 2.2. For $m \in N$ and sufficiently large $n \in N$, there holds the following recurrence relation

$$B_n\left((t-x)^m, k, x\right) = n^{-(k+1)}\left[Q(m,k,x) + o(1)\right]$$

where $Q(m,k,x)$ is a certain polynomial in x of degree m and $x \in [0,\infty)$ is arbitrary but fixed.

Proof. Using Lemma 2.1, for sufficiently large n, we can write

$$B_n\left((t-x)^m, x\right) = \frac{q_0(x)}{n^{[(m+1)/2]}} + \frac{q_1(x)}{n^{[(m+1)/2]+1}} + \ldots + \frac{q_{[m/2]}(x)}{n^m} + \ldots$$

where $q_i(x)$, $i = 0$, 1, 2, 3, … are certain polynomials in x of degree at most m.

Now, we have

$$B_n\left((t-x)^m, k, x\right) = \frac{1}{\Delta}\begin{vmatrix} \left(\frac{q_0(x)}{d_0 n^{[(m+1)/2]}} + \frac{q_1(x)}{d_0 n^{[(m+1)/2]+1}} + \ldots\right) & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k} \\ \left(\frac{q_0(x)}{d_1 n^{[(m+1)/2]}} + \frac{q_1(x)}{d_1 n^{[(m+1)/2]+1}} + \ldots\right) & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots & \ldots & \ldots \\ \left(\frac{q_0(x)}{d_k n^{[(m+1)/2]}} + \frac{q_1(x)}{d_k n^{[(m+1)/2]+1}} + \ldots\right) & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k} \end{vmatrix}$$

$$= n^{-(k+1)}\left[Q(m,k,x) + o(1) \right]$$, for each fixed $x \in [0,\infty)$.

This completes the proof of the Lemma 2.2.

Lemma 2.3. For $m \in N^0$, if the m[th] order moment for the operators (1.1) be defined as

$$U_{n,m}(x) = \frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\left(\frac{v-1}{n+2} - x\right)^m,$$

then, we have $U_{n,0}(x) = 1, U_{n,1}(x) = 0$ and there holds the recurrence relation

$$(n+2)U_{n,m+1}(x) = x(1+x)[U'_{n,m}(x) + mU_{n,m-1}(x)].$$

Consequently, for each $x \in [0,\infty)$, we have from this relation that

$$U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right),$$

where [α] denotes the integral part of α .

Proof. Using the definition of $U_{n,m}(x)$ and basic properties of $b_{n,v}(x)$, we obtain

$$U_{n,0}(x) = 1 \quad \text{and} \quad U_{n,1}(x) = 0.$$

Now, we have

$$x(1+x)U'_{n,m}(x) = \frac{1}{(n+1)}\sum_{v=1}^{\infty} x(1+x)b'_{n,v}(x)\left(\frac{v-1}{n+2} - x\right)^m$$

$$-x(1+x)\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)m\left(\frac{v-1}{n+2}-x\right)^{m-1}$$

Thus using basic properties of $b_{n,v}(x)$, we get

$$\begin{aligned}x(1+x)\left[U'_{n,m}(x)+mU_{n,m-1}(x)\right]&=\frac{1}{(n+1)}\sum_{v=1}^{\infty}[(v-1)-(n+2)x]b_{n,v}(x)\left(\frac{v-1}{n+2}-x\right)^{m}\\&=\frac{(n+2)}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\left(\frac{v-1}{n+2}-x\right)^{m+1}\\&=(n+2)U_{n,m+1}(x)\end{aligned}$$

This completes the proof of the recurrence relation.

The other consequence follows easily from the recurrence relation.

Lemma 2.4([6]). There exist the polynomials $Q_{i,j,r}(x)$ independent of n and v such that

$$[x(1+x)]^{r}D^{r}b_{n,v}(x)=\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+2)^{i}[(v-1)-(n+2)x]^{j}Q_{i,j,r}(x)b_{n,v}(x),$$

where D^{r} is the r^{th} order differentiation operator.

Lemma 2.5. If C(j, k) , j = 0, 1, 2, …, k are defined as in (1.3), then we have

$$\sum_{j=0}^{k}C(j,k)d_j^{-m}=\begin{cases}1 & \text{for } m=0\\ 0 & \text{for } m=1,2,\ldots,k\end{cases}.$$

Proof. From relations (1.2) and (1.3) we get

$$\sum_{j=0}^{k}C(j,k)d_j^{-m}=\begin{vmatrix}d_0^{-m} & d_0^{-1} & d_0^{-2} & \cdots\cdots & d_0^{-k}\\ d_1^{-m} & d_1^{-1} & d_1^{-2} & \cdots\cdots & d_1^{-k}\\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots\\ d_k^{-m} & d_k^{-1} & d_k^{-2} & \cdots\cdots & d_k^{-k}\end{vmatrix}\Bigg/\begin{vmatrix}1 & d_0^{-1} & d_0^{-2} & \cdots\cdots & d_0^{-k}\\ 1 & d_1^{-1} & d_1^{-2} & \cdots\cdots & d_1^{-k}\\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots\\ 1 & d_k^{-1} & d_k^{-2} & \cdots\cdots & d_k^{-k}\end{vmatrix}$$

which implies that

$$\sum_{j=0}^{k}C(j,k)d_j^{-m}=\begin{cases}1 & \text{for } m=0\\ 0 & \text{for } m=1,2,\ldots,k\end{cases}.$$

III. MAIN RESULT

Now we begin to prove the main results of this section, namely, Voronovskaja-type asymptotic formula.

Theorem: Let $r \in N$. If the function $f \in H[0,\infty)$ is bounded on every finite subinterval of $[0,\infty)$ admitting a derivative of order (2k+r+2) at a fixed point $x \in (0,\infty)$, satisfying $f(t) = O(t^{\gamma})$ as $t \to \infty$ for some $\gamma > 0$, then we have

$$\lim_{n \to \infty} n^{k+1}[B_n^{(r)}(f,k,x) - f^{(r)}(x)] = \sum_{i=r+1}^{2k+r+2} f^{(i)}(x) Q(i,k,r,x)$$

(3.1)

and

$$\lim_{n \to \infty} n^{k+1}[B_n^{(r)}(f,k+1,x) - f^{(r)}(x)] = 0,$$

(3.2)

where $Q(i,k,r,x)$ are certain polynomials in x.

Moreover, if $f^{(2k+r+2)}$ exists and is continuous on $(a-\Delta, b+\Delta) \subset (0,\infty)$ where $\Delta > 0$, then (3.1) and (3.2) hold uniformly on $[a,b]$.

Proof. Since $f^{(2k+r+2)}$ exists at $x \in (0,\infty)$, therefore by Taylor's expansion of f (t), we have

$$f(t) = \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{2k+r+2}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x) = O\left((t-x)^{\gamma}\right)$ as $t \to \infty$.

Using linearity of $B_n^{(r)}(\cdot, k, x)$, relation (1.3) and the above expansion of f (t), we get

$$n^{k+1}[B_n^{(r)}(f(t),k,x) - f^{(r)}(x)] = n^{k+1}\left[\sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!} B_n^{(r)}\left((t-x)^i,k,x\right) - f^{(r)}(x)\right]$$

$$+ n^{k+1}\sum_{j=0}^{k} C(j,k)\frac{1}{(d_j n+1)}\sum_{v=1}^{\infty} b_{d_j n,v}^{(r)}(x)\int_0^{\infty} b_{d_j n,v}(t)\varepsilon(t,x)\,(t-x)^{2k+r+2}\,dt$$

$=J_1+J_2$ (say)

Using Lemma 2.2, we obtain

$$J_1 = n^{k+1}\left[\sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!} B_n^{(r)}\left((t-x)^i,k,x\right) - f^{(r)}(x)\right]$$

$$= n^{k+1}\left[f^{(r)}(x) + \sum_{i=r+1}^{2k+r+2} \frac{f^{(i)}(x)}{i!} D^r\left(n^{-(k+1)}\left[Q(i,k,x) + o(1)\right]\right) - f^{(r)}(x)\right]$$

$$= \sum_{i=r+1}^{2k+r+2} f^{(i)}(x) Q(i,k,r,x) + o(1)$$

In order to prove assertion (3.1), it is enough to show that $J_2 \to 0$ as $n \to \infty$.

Let $$J_{21}=n^{k+1}\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}^{(r)}(x)\int_0^{\infty}b_{n,v}(t)\,\varepsilon(t,x)(t-x)^{2k+r+2}\,dt\,.$$

Then, using Lemma 2.4 and Schwarz inequality for summation, we obtain

$$\left|J_{21}\right|\le n^{k+1}\sum_{v=1}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}\frac{(n+2)^i}{(n+1)}\left|(v-1)-(n+2)x\right|^j\frac{\left|Q_{i,j,r}(x)\right|}{[x(1+x)]^r}b_{n,v}(x)$$

$$\times\int_0^{\infty}b_{n,v}(t)\left|\varepsilon(t,x)\right|\,\left|t-x\right|^{2k+r+2}dt$$

$$\le n^{k+1}C(x)\sum_{\substack{2i+j\le r\\ i,j\ge 0}}\frac{(n+2)^i}{(n+1)}\sum_{v=1}^{\infty}\left|(v-1)-(n+2)x\right|^j b_{n,v}(x)\int_0^{\infty}b_{n,v}(t)\left|\varepsilon(t,x)\right|\,\left|t-x\right|^{2k+r+2}dt$$

$$\le n^{k+1}C(x)\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+2)^i\left(\frac{1}{(n+1)}\sum_{v=1}^{\infty}\left[(v-1)-(n+2)x\right]^{2j}b_{n,v}(x)\right)^{1/2}$$

$$\times\left(\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\left[\int_0^{\infty}b_{n,v}(t)\left|\varepsilon(t,x)\right|\,\left|t-x\right|^{2k+r+2}dt\right]^2\right)^{1/2},$$

where $$C(x)=\sup_{\substack{2i+j\le r\\ i,j\ge 0}}\ \sup_{x\in[a,b]}\frac{\left|Q_{i,j,r}(x)\right|}{[x(1+x)]^r}.$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$, therefore, for a given $\varepsilon>0$ there exists a $\delta>0$ such that

$$\left|\varepsilon(t,x)\right|<\varepsilon \text{ whenever } 0<\left|t-x\right|<\delta\,.$$

Also, since $\varepsilon(t,x)=O\left((t-x)^{\gamma}\right)$, therefore there exists a positive constant C_1 such that

$$\left|\varepsilon(t,x)\right|=C_1\left|t-x\right|^{\gamma} \text{ for all } \left|t-x\right|\ge\delta\,.$$

Now applying Schwarz inequality for integration and Lemma 2.1, we have

$$J_{211}=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\left[\int_0^{\infty}b_{n,v}(t)\left|\varepsilon(t,x)\right|\,\left|t-x\right|^{2k+r+2}dt\right]^2$$

$$\le\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\left(\int_0^{\infty}b_{n,v}(t)\,dt\right)\left(\int_0^{\infty}b_{n,v}(t)\left[\varepsilon(t,x)\right]^2(t-x)^{4k+2r+4}dt\right)$$

$$=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\left(\int_{|t-x|<\delta}+\int_{|t-x|\ge\delta}\right)b_{n,v}(t)\left[\varepsilon(t,x)\right]^2(t-x)^{4k+2r+4}dt$$

$$\leq \frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\left(\varepsilon^2 \int_{|t-x|<\delta} b_{n,v}(t)(t-x)^{4k+2r+4}\,dt + C_1^2 \int_{|t-x|\geq\delta} b_{n,v}(t)(t-x)^{4k+2r+2\gamma+4}\,dt\right)$$

$$\leq \frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\left(\varepsilon^2 \int_0^{\infty} b_{n,v}(t)(t-x)^{4k+2r+4}\,dt + \frac{C_1^2}{\delta^{2q-(4k+2r+2\gamma+4)}} \int_0^{\infty} b_{n,v}(t)(t-x)^{2q}\,dt\right)$$

$$= \varepsilon^2 O\left(n^{-(2k+r+2)}\right) + C_2 O(n^{-q}),$$

where q is a positive constant and the constant C_2 is dependent on q, k, r, δ and γ .

Next using Lemma 2.3, we get

$$|J_{21}| \leq n^{k+1} C(x) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+2)^{i+j} O(n^{-j/2})\left[\varepsilon^2 O\left(n^{-(2k+r+2)}\right) + C_2 O(n^{-q})\right]^{1/2}$$

$$= C(x) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} O\left(n^{(2i+j+2k+2)/2}\right)\left[\varepsilon^2 O\left(n^{-(2k+r+2)}\right) + C_2 O(n^{-q})\right]^{1/2}$$

$$\leq C(x) O\left(n^{(2k+r+2)/2}\right)\left[\varepsilon^2 O\left(n^{-(2k+r+2)}\right) + C_2 O(n^{-q})\right]^{1/2}$$

$$= C(x)\left[\varepsilon^2 O(1) + C_2 O(n^{2k+r+2-q})\right]^{1/2} = C(x)\,\varepsilon\, O(1),$$

where q is chosen to be greater than 2k + r + 2.

Thus, due to arbitrariness of $\varepsilon > 0$ and by Lemma 2.5 it follows that $J_2 \to 0$ as $n \to \infty$.

Finally collecting the estimates of J_1 and J_2, assertion (3.1) follows.

The assertion (3.2) can be proved easily in a similar manner noticing the fact that

$$B_n\left((t-x)^i, k+1, x\right) = O\left(n^{-(k+2)}\right) \quad \text{for } i = 1, 2, \ldots$$

The last uniformity assertion follows from the fact that $\delta(\varepsilon)$ in the proof of the assertion (3.1) can chosen to be independent of $x \in [a,b]$ and rest of the estimates hold uniformly on $[a,b]$.

This completes the proof of the theorem.

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