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Common Fixed Point Theorems in Dislocated Quasi *b*-Metric Spaces Satisfying Contractive Condition of Integral Type

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ABSTRACT

In this paper, common fixed point theorems is proved in dislocated quasi b -metric spaces. Our result generalized, modified, some existing result in the literature.

Keywords. Dislocated quasi b -metric space, Cauchy sequence , Common fixed point.

I. INTRODUCTION AND PREILMANARIES

Frechet[5] introduced the notion of metric space in 1906. Hitzler and Seda[8] introduced the notion of dislocated metric spaces. Zeyada et al.[3] generalized the result of Hitzler and Seda[8] and introduced the concept of complete dislocated quasi metric space. In 1989, Bakhtin[4] introduced the b -metric space as a generalization of metric space and investigated some fixed point theorem in such spaces. The concept of quasi b -metric spaces given by Shah and Huassain[6] in 2012 and obtained some fixed point results. Chakkrid and Cholotis[2] introduced the concept of dislocated quasi b -metric spaces. Recently Mujeeb Ur Rahman and Muhammad Sarwar[7] define the notion of coupled coincidence fixed point and proved a coupled coincidence fixed point theorem in dislocated quasi b-metric space. Aage and Golhare[1] proved common fixed point theorem in dislocated quasi b-metric space.

In this paper, common fixed point theorem is proved in dislocated quasi b-metric space satisfying contractive condition of integral type.

Definition 1.1[3&8] Let X be a non empty set and let $d: X \times X \to [0, \infty)$ be a function satisfying the following conditions:

(d_1) $d(x, x) = 0$,

(d_2) $d(x, y)=d(y, x)=0$ implies $x = y$,

(d_3) $d(x, y)= d(y, x)$ for all $x, y \in X$,

(d_4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfies conditions only (d_2) and (d_4), then d is called a dislocated quasi metric on X.

If d satisfies conditions (d_1), (d_2) and (d_4) then d is called a quasi metric on X. If d satisfies conditions (d_2), (d_3) and (d_4) then d is called a dislocated metric on X .If d satisfies all the conditions (d_1), (d_2), (d_3) and (d_4) then d is called a metric on X.

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Definition 1.2[4]. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to [0, \infty)$ is called a b -metric if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is called a b -metric space. The number $s \geq 1$ is called the coefficient of (X, d).

Definition 1.3.[6] Let X be a non-empty set. Let $d: X \times X \to [0, \infty)$ be a mapping and $s \geq 1$ be a constant satisfy the following conditions

(i) $d(x, y) = 0 = d(y, x)$ iff $x = y$, for all $x, y \in X$,

(ii) $d(x, y) \leq s[d(x, z) + d(z, y)]$, for all $x, y, z \in X$.

Then pair (X, d) is called quasi b -metric space.

Definition 1.4[2]. Let X be a non-empty set. Let the mapping $d: X \times X \to [0, \infty)$ and constant $s \geq 1$ satisfy following conditions:

(i) $d(x, y) = 0 = d(y, x) \Rightarrow x = y$, for all $x, y \in X$,

(ii) $d(x, y) \leq s[d(x, z) + d(z, y)]$, for all $x, y, z \in X$.

Then the pair (X, d) is called dislocated quasi b -metric space or in short dq b -metric space.

Example 1.1. Let $X = \mathbb{R}$ and suppose

$$d(x, y) = |2x - y|^2 + |2x + y|^2$$

Then (X, d) is a dislocated quasi b-metric space with the coeficient $s = 2$. But it is not dislocated quasi-metric space nor b-metric space.

Definition 1.5[2]. Let (X, d) be a dq b -metric space. A sequence $\{x_n\}$ in X is called to be dq b -converges to $x \in X$ if

$$\lim_{n \to \infty} d(x_n, x) = 0 = \lim_{n \to \infty} d(x, x_n)$$

In this case x is called dq b -limit of $\{x_n\}$ and is written as $x_n \to x$.

Definition 1.6[2]. Let (X, d) be a dq b -metric space. A sequence $\{x_n\}$ in X is called dq b -Cauchy sequence if

$$\lim_{n,m \to \infty} d(x_n, x_m) = 0 = \lim_{n,m \to \infty} d(x_m, x_n)$$

Definition 1.7[2]. A dq b -metric space (X, d) is said to be dq b -complete if every dq b -Cauchy sequence in it is dq b -convergent in X.

Proposition 1.1[2]. Every subsequence of a dq b -convergent sequence in a dq b -metric space (X, d) is dq b -convergent sequence.

Proposition 1.2[2]. Every subsequence of a dq b -Cauchy sequence in a dq b -metric space (X, d) is dq b -Cauchy sequence.metric space.

Lemma 1.1. Limit of a convergent sequence in dislocated quasi b -metric space is unique.

Lemma 1.2. Let (X; d) be a dislocated quasi b -metric space and $\{x_n\}$ be a sequence in dq b -metric space such that

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$$d(x_n, x_{n+1}) \leq \alpha\, d(x_{n-1}, x_n)$$

for $n = 1,2,3, \ldots$, $0 \leq \alpha s < 1$, $\alpha \in [0,1)$ and s is defined in dq b -metric space. Then $\{x_n\}$ is a Cauchy sequence in X.

II. MAIN RESULT.

Theorem 2.1. Let (X, d) be a complete dislocated quasi b-metric space and let $f, g: X \to X$ be a self mappings on X. For $s \geq 1$ satisfying :

(2.1.1) $$\int_0^{d(fx,gy)} \varphi(t)dt \leq \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right)$$

where

$$M(x, y) = \lambda d(x, y) + \mu\left\{\frac{d(x,fx)d(y,gy)}{1+d(x,y)}\right\} + \gamma\left\{\frac{d(y,fx)d(x,gy)}{1+d(x,y)}\right\}$$ and $\varphi \in \Phi$, $\psi \in \Psi$.

For all $x, y \in X$ such that $1 + d(x, y) \neq 0$ and λ, μ, γ are non-negative reals with $\lambda + \mu < 1$. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X and define

$$x_{n+1} = fx_n \quad \text{and} \quad x_{n+2} = gx_{n+1}, \quad n = 0,1,2, \ldots .$$

Consider

$$\int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt = \int_0^{d(fx_n,gx_{n+1})} \varphi(t)dt$$

$$\leq \psi\left(\int_0^{M(x_n,x_{n+1})} \varphi(t)dt\right) \qquad (2.1.2)$$

where

$$M(x_n, x_{n+1}) \leq \lambda d(x_n, x_{n+1}) + \mu\left\{\frac{d(x_n,fx_n)d(x_{n+1},gx_{n+1})}{1+d(x_n,x_{n+1})}\right\} + \gamma\left\{\frac{d(x_{n+1},fx_n)d(x_n,gx_{n+1})}{1+d(x_n,x_{n+1})}\right\}$$

$$\leq \lambda d(x_n, x_{n+1}) + \mu\left\{\frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})}\right\} + \gamma\left\{\frac{d(x_{n+1},x_{n+1})d(x_n,x_{n+2})}{1+d(x_n,x_{n+1})}\right\}$$

But $1 + d(x_n, x_{n+1}) > d(x_n, x_{n+1})$

or $\frac{d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})} < 1.$

Thus, we have

$$M(x_n, x_{n+1}) < \lambda d(x_n, x_{n+1}) + \mu\, d(x_{n+1}, x_{n+2})$$

that is $d(x_{n+1}, x_{n+2}) < \lambda d(x_n, x_{n+1}) + \mu\, d(x_{n+1}, x_{n+2})$

$$(1 - \mu)d(x_{n+1}, x_{n+2}) < \lambda d(x_n, x_{n+1})$$

$d(x_{n+1},x_{n+2}) < \frac{\lambda}{(1-\mu)} d(x_n,x_{n+1})$

$d(x_{n+1},x_{n+2}) < k\, d(x_n,x_{n+1})$, where $k = \frac{\lambda}{(1-\mu)} < 1$.

Hence by (2.1.2), we have

$$\int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt \leq \psi\left(\int_0^{k\, d(x_n,x_{n+1})} \varphi(t)dt\right).$$

If $k < 1$ then by inductivity, we obtain

$$\int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt \leq \psi\left(\int_0^{k\, d(x_n,x_{n+1})} \varphi(t)dt\right) \leq \psi\left(\int_0^{k^2\, d(x_{n-1},x_n)} \varphi(t)dt\right)$$

$$\leq \cdots \leq \psi\left(\int_0^{k^{n+1}\, d(x_0,x_1)} \varphi(t)dt\right).$$

So that for any $m > n$, we get

$$\int_0^{d(x_n,x_m)} \varphi(t)dt = \int_0^{sd(x_n,x_{n+1})+sd(x_{n+1},x_m)} \varphi(t)dt$$

$$= \int_0^{sd(x_n,x_{n+1})+s^2 d(x_{n+1},x_{n+2})+s^2 d(x_{n+2},x_m)} \varphi(t)dt$$

$$= \int_0^{sd(x_n,x_{n+1})+s^2 d(x_{n+1},x_{n+2})+s^3 d(x_{n+2},x_{n+3})+\cdots+s^{m-n} d(x_{m-1},x_m)} \varphi(t)dt$$

$$\leq \psi\left(\int_0^{sk^n d(x_0,x_1)+s^2 k^{n+1} d(x_0,x_1)+s^3 k^{n+2} d(x_0,x_1)+\cdots+s^{m-n} k^{m-1} d(x_0,x_1)} \varphi(t)dt\right)$$

$$\leq \psi\left(\int_0^{(sk^n+s^2 k^{n+1}+s^3 k^{n+2}+\cdots+s^{m-n} k^{m-1}) d(x_0,x_1)} \varphi(t)dt\right)$$

$$\leq \psi\left(\int_0^{\frac{sk^n}{1-sk}} \varphi(t)dt\right).$$

Since $sk, k < 1$, we have

$$\int_0^{d(x_n,x_m)} \varphi(t)dt \leq \psi\left(\int_0^{\frac{sk^n}{1-sk}} \varphi(t)dt\right) \to 0 \text{ as } n,m \to \infty.$$

Thus $\{x_n\}$ is a dq b-Cauchy sequence in X. If X is dq b-complete, there exist some $t \in X$ such that $x_n \to t$ as $n \to \infty$. Now we prove that $ft = t$. Suppose if not, there exists $u \in X$ such that

$$d(t,ft) = u > 0.$$

Consider

$$\int_0^u \varphi(t)dt = \int_0^{d(t,ft)} \varphi(t)dt$$

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$$= \int_0^{sd(t,x_{n+2})+sd(x_{n+2},ft)} \varphi(t)dt$$

$$= \int_0^{sd(t,x_{n+2})+sd(gx_{n+1},ft)} \varphi(t)dt$$

$$\int_0^u \varphi(t)dt \le \psi\left(\int_0^{sd(t,x_{n+2})+s\left[\lambda d(t,x_{n+1})+\mu\left\{\frac{d(t,ft)d(x_{n+1},gx_{n+1})}{1+d(t,x_{n+1})}\right\}+\gamma\left\{\frac{d(x_{n+1},ft)d(t,gx_{n+1})}{1+d(t,x_{n+1})}\right\}\right]} \varphi(t)dt\right).$$

Letting $n \to \infty$, we get $u = d(t, ft) = 0$, which is a contradiction so that $u = 0$.

Hence $ft = t$.

Similarly, we can show that $gt = t$.

To prove the uniqueness of common fixed point of f and g. Suppose that $v(\neq t)$ be another common fixed point of f and g. Then

$$\int_0^{d(t,v)} \varphi(t)dt = \int_0^{d(ft,gv)} \varphi(t)dt \le \psi\left(\int_0^{M(t,v)} \varphi(t)dt\right),$$

where

$$M(t,v) = \lambda d(t,v) + \mu\left\{\frac{d(t,ft)d(v,gv)}{1+d(t,v)}\right\} + \gamma\left\{\frac{d(v,ft)d(t,gv)}{1+d(t,v)}\right\}$$

$$= \lambda d(t,v) + \mu\left\{\frac{d(t,t)d(v,v)}{1+d(t,v)}\right\} + \gamma\left\{\frac{d(v,t)d(t,v)}{1+d(t,v)}\right\}$$

$$< \lambda d(t,v) + \gamma\, d(t,v) = (\lambda+\gamma)d(t,v).$$

That is

$\int_0^{d(t,v)} \varphi(t)dt \le \psi\left(\int_0^{(\lambda+\gamma)d(t,v)} \varphi(t)dt\right)$, which is a contradiction. So $t = v$.

Hence t is unique common fixed point of f and g.

Remarks 2.1. By setting $f = g$ in theorem 2.1, we get the following result.

Theorem 2.2. Let (X, d) be a complete dislocated quasi b-metric space and let $f: X \to X$ be a self mappings on X. For $s \ge 1$ satisfying :

(2.2.1) $$\int_0^{d(fx,fy)} \varphi(t)dt \le \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right)$$

where

$M(x,y) = \lambda d(x,y) + \mu\left\{\frac{d(x,fx)d(y,fy)}{1+d(x,y)}\right\} + \gamma\left\{\frac{d(y,fx)d(x,fy)}{1+d(x,y)}\right\}$ and $\varphi \in \Phi, \psi \in \Psi$.

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For all $x, y \in X$ such that $1 + d(x, y) \neq 0$ and λ, μ, γ are non-negative reals with $\lambda + \mu < 1$. Then f has a unique common fixed point.

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