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Polynomials having no zeros in an Open Disc

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ABSTRACT

In this paper we put certain restrictions on the coefficients and also onthe real and imaginary of the coefficients of a polynomial and find open discs which contain no zero of the polynomial.

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Key Words and Phrases: ***Coefficient, Open Disc, Polynomial, Zeros.***

I. INTRODUCTION

The study of polynomials and the location of their zeros play an important role in many areas of the discipline of science such as communication theory, signal processing, control theory, cryptography, coding theory, mathematical biology, combinatorics, chemistry, graph theory etc. Though the fundamental theorem of algebra guarantees the existence of exactly n zeros of a polynomial of degree n, there is no method available to find these zeros. Therefore, there is need to put some conditions on the coefficients of a polynomial and locate the regions that contain all or some of the zeros of the polynomial. In this paper, we subject the coefficients and/or the real and imaginary parts of a polynomial to certain restrictions and determine open discs which contain no zeros of the polynomial. A classical result regarding the location of zeros of a polynomial with real coefficients is the following known as the Enestrom-Kakeya Theorem [4,5]:

Theorem A: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ......\ge a_1 \ge a_0 > 0 .$$

Then all the zeros of P(z) lie in $|z| \le 1$.

Several generalizations, extensions and refinements of the above result are available in the literature.

Recently Gulzar et al [3] after making corrections in a paper of K.A.Kareem and A.A.Mogbademu [2] on polynomials with certain monotonicity conditions on their coefficients , proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n where for some R>0,

$$0 \le \mu < 1, 0 < \rho \le 1, 0 \le k \le n,$$

$$0 < \rho|a_0| \le R|a_1| \le R^2|a_2| \le .... \le R^{k-1}|a_{k-1}| \le R^k|a_k|$$

$$\ge R^{k+1}|a_{k+1}| \ge ...... \ge R^{n-1}|a_{n+1}| \ge (R-\mu)R^{n-1}|a_n| ,$$

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and $\left|\arg a_j - \beta\right| \le \alpha \le \frac{\pi}{2}$ for $1 \le j \le n$ and for some real α and β. Then for $0 < \delta < 1$ the number of zeros of P(z) in $|z| \le \delta R$ is less than

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|}$$

where

$$M = 2|a_0|R - \rho|a_0|[1 - \cos\alpha - \sin\alpha] + 2|a_k|R^{k+1}\cos\alpha$$
$$+ |a_n|R^{n+1}[1 - \cos\alpha + \sin\alpha] + \mu|a_n|R^n(1 + \cos\alpha - \sin\alpha) + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|R^{j+1}.$$

Theorem C: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$ for $0 \le j \le n$. Suppose that for some R>0, $0 \le \mu < 1, 0 < \rho \le 1, 0 \le k \le n,$

$$0 \ne \rho\alpha_0 \le R\alpha_1 \le R^2\alpha_2 \le .... \le R^{k-1}\alpha_{k-1} \le R^k\alpha_k$$
$$\ge R^{k+1}\alpha_{k+1} \ge ...... \ge R^{n-1}\alpha_{n+1} \ge (R-\mu)R^{n-1}\alpha_n .$$

Then for $0 < \delta < 1$ the number of zeros of P(z) in $|z| \le \delta R$ is less than

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|}$$

where

$$M = 2|\alpha_0|R - \rho(\alpha_0 + |\alpha_0|)R + (|\alpha_n| - \alpha_n)R^{n+1} + \mu(\alpha_n + |\alpha_n|)R^n$$
$$+ 2\alpha_k R^{k+1} + 2\sum_{j=0}^{n}|\beta_j|R^{j+1}.$$

Theorem D: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where

$\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$ for $0 \le j \le n$. Suppose that for some

R>0, $0 \le \mu < 1, 0 \le \lambda < 1, 0 < \rho_1 \le 1, 0 < \rho_2 \le 1$ and for some $0 \le k \le n, 0 \le l \le n,$

$$0 \ne \rho_1\alpha_0 \le R\alpha_1 \le R^2\alpha_2 \le .... \le R^{k-1}\alpha_{k-1} \le R^k\alpha_k$$
$$\ge R^{k+1}\alpha_{k+1} \ge ...... \ge R^{n-1}\alpha_{n+1} \ge (R-\mu)R^{n-1}\alpha_n$$

and

$$\rho_2\beta_0 \le R\beta_1 \le R^2\beta_2 \le .... \le R^{l-1}\beta_{l-1} \le R^l\beta_l$$

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$$\geq R^{l+1}\beta_{l+1} \geq ...... \geq R^{n-1}\beta_{n+1} \geq (R-\lambda)R^{n-1}\beta_n.$$

Then for $0<\delta<1$ the number of zeros of P(z) in the disc $|z|\leq \delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$M = 2|\alpha_0|R-\rho_1 R(|\alpha_0|+\alpha_0)+2\alpha_k R^{k+1}+(|\alpha_n|-\alpha_n)R^{n+1}+\mu(\alpha_n+|\alpha_n|)R^n$$
$$+2|\beta_0|R-\rho_2 R(|\beta_0|+\beta_0)+2\beta_l R^{l+1}+(|\beta_n|-\beta_n)R^{n+1}+\lambda(\beta_n+|\beta_n|)R^n$$

II. MAIN RESULTS

The aim of this paper is to find open discs which do not contain any zeros of the polynomials considered in the above mentioned results and to prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where for some R>0,

$$0\leq \mu<1, 0<\rho\leq 1, 0\leq k\leq n,$$
$$0<\rho|a_0|\leq R|a_1|\leq R^2|a_2|\leq .... \leq R^{k-1}|a_{k-1}|\leq R^k|a_k|$$
$$\geq R^{k+1}|a_{k+1}|\geq ......\geq R^{n-1}|a_{n+1}|\geq (R-\mu)R^{n-1}|a_n|,$$

and $|\arg a_j-\beta|\leq \alpha\leq \frac{\pi}{2}$ for $1\leq j\leq n$ and for some real α and β. Then

P(z) has no zero in $|z|<\frac{|a_0|R}{M'}$ where

$$M' = |a_0|R-\rho|a_0|R(1+\cos\alpha-\sin\alpha)+2|a_k|R^{k+1}\cos\alpha$$
$$+|a_n|R^{n+1}(1+\sin\alpha-\cos\alpha)+\mu|a_n|R^n(1+\cos\alpha-\sin\alpha)$$
$$+2\sin\alpha\sum_{j=0}^{n-1}|a_j|R^{j+1}$$

Taking R=1 in Theorem 1, we get the following

Corollary 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where for some

$$0\leq \mu<1, 0<\rho\leq 1, 0\leq k\leq n,$$
$$0<\rho|a_0|\leq |a_1|\leq |a_2|\leq .... \leq |a_{k-1}|\leq |a_k|$$
$$\geq |a_{k+1}|\geq ......\geq |a_{n+1}|\geq (1-\mu)|a_n|,$$

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and $\left|\arg a_j - \beta\right| \le \alpha \le \dfrac{\pi}{2}$ for $1 \le j \le n$ and for some real α and β. Then

P(z) has no zero in $|z| < \dfrac{|a_0|}{M'}$ where

$$M' = |a_0| - \rho|a_0|(1+\cos\alpha - \sin\alpha) + 2|a_k|\cos\alpha$$
$$+ |a_n|(1+\sin\alpha - \cos\alpha) + \mu|a_n|(1+\cos\alpha - \sin\alpha)$$
$$+ 2\sin\alpha\sum_{j=0}^{n-1}|a_j|.$$

Theorem 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where $\operatorname{Re}(a_j) = \alpha_j$ and $\operatorname{Im}(a_j) = \beta_j$ for $0 \le j \le n$. Suppose that for some R>0, $0 \le \mu < 1, 0 < \rho \le 1, 0 \le k \le n,$

$$0 \ne \rho\alpha_0 \le R\alpha_1 \le R^2\alpha_2 \le \dots\dots\dots \le R^{k-1}\alpha_{k-1} \le R^k\alpha_k$$
$$\ge R^{k+1}\alpha_{k+1} \ge \dots\dots \ge R^{n-1}\alpha_{n+1} \ge (R-\mu)R^{n-1}\alpha_n.$$

Then P(z) has no zero in $|z| < \dfrac{|a_0|R}{M'}$ where

$$M' = |\alpha_0|R - \rho R(|\alpha_0| + \alpha_0) + 2\alpha_k R^{k+1} + (|\alpha_n| - \alpha_n)R^{n+1} + \mu R^n(|\alpha_n| + \alpha_n)$$
$$+ 2\sum_{j=0}^{n}|\beta_j|R^{j+1}$$

Taking R=1 in Theorem 2, we get the following

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where $\operatorname{Re}(a_j) = \alpha_j$ and $\operatorname{Im}(a_j) = \beta_j$ for $0 \le j \le n$. Suppose that for some $0 \le \mu < 1, 0 < \rho \le 1, 0 \le k \le n,$

$$0 \ne \rho\alpha_0 \le \alpha_1 \le \alpha_2 \le \dots\dots\dots \le \alpha_{k-1} \le \alpha_k$$
$$\ge \alpha_{k+1} \ge \dots\dots \ge \alpha_{n+1} \ge (1-\mu)\alpha_n.$$

Then P(z) has no zero in $|z| < \dfrac{|a_0|}{M'}$ where

$$M' = |\alpha_0| - \rho(|\alpha_0| + \alpha_0) + 2\alpha_k + (|\alpha_n| - \alpha_n) + \mu(|\alpha_n| + \alpha_n)$$
$$+ 2\sum_{j=0}^{n}|\beta_j|$$

Theorem 3: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where

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$\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$ for $0 \le j \le n$. Suppose that for some

R>0, $0 \le \mu < 1, 0 \le \lambda < 1, 0 < \rho_1 \le 1, 0 < \rho_2 \le 1$ and for some $0 \le k \le n, 0 \le l \le n,$

$$0 \ne \rho_1 \alpha_0 \le R\alpha_1 \le R^2 \alpha_2 \le .... \le R^{k-1}\alpha_{k-1} \le R^k \alpha_k$$
$$\ge R^{k+1}\alpha_{k+1} \ge ...... \ge R^{n-1}\alpha_{n+1} \ge (R-\mu)R^{n-1}\alpha_n$$

and

$$\rho_2 \beta_0 \le R\beta_1 \le R^2 \beta_2 \le .... \le R^{l-1}\beta_{l-1} \le R^l \beta_l$$
$$\ge R^{l+1}\beta_{l+1} \ge ...... \ge R^{n-1}\beta_{n+1} \ge (R-\mu)R^{n-1}\beta_n.$$

Then P(z) has no zero in $|z| < \frac{|a_0|R}{M'}$ where

$$M' = |\alpha_0|R - \rho_1 R(|\alpha_0| + \alpha_0) + 2\alpha_k R^{k+1} + (|\alpha_n| - \alpha_n)R^{n+1} + \mu R^n(|\alpha_n| + \alpha_n)$$
$$+ |\beta_0|R - \rho_2 R(|\beta_0| + \beta_0) + 2\beta_l R^{l+1} + (|\beta_n| - \beta_n)R^{n+1} + \lambda R^n(|\beta_n| + \beta_n)$$

Taking R=1, $\lambda = \mu, \rho_1 = \rho_2 = \rho, l = k$ in Theorem 3, we get the following

Corollary 3: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where

$\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$ for $0 \le j \le n$. Suppose that for some $0 \le \mu < 1, 0 < \rho \le 1$ and for some $0 \le k \le n,$

$$0 \ne \rho\alpha_0 \le \alpha_1 \le \alpha_2 \le .... \le \alpha_{k-1} \le \alpha_k$$
$$\ge \alpha_{k+1} \ge ...... \ge \alpha_{n+1} \ge (1-\mu)\alpha_n$$

and

$$\rho\beta_0 \le \beta_1 \le \beta_2 \le .... \le \beta_{k-1} \le \beta_k$$
$$\ge \beta_{k+1} \ge ...... \ge \beta_{n+1} \ge (1-\mu)\beta_n.$$

Then P(z) has no zero in $|z| < \frac{|a_0|}{M'}$ where

$$M' = |\alpha_0| - \rho_1(|\alpha_0| + \alpha_0) + 2\alpha_k + (|\alpha_n| - \alpha_n) + \mu(|\alpha_n| + \alpha_n)$$
$$+ |\beta_0| - \rho_2(|\beta_0| + \beta_0) + 2\beta_k + (|\beta_n| - \beta_n) + \lambda(|\beta_n| + \beta_n).$$

Combining Theorem 1 and Theorem B, we get the following corollary giving a bound for the number of zeros of P(z) in an annular region:

Corollary 4: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where for some

R>0, $0 \le \mu < 1, 0 < \rho \le 1, 0 \le k \le n,$

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$$0<\rho|a_0|\le R|a_1|\le R^2|a_2|\le \dots\dots\dots \le R^{k-1}|a_{k-1}|\le R^k|a_k|$$
$$\ge R^{k+1}|a_{k+1}|\ge \dots\dots \ge R^{n-1}|a_{n+1}|\ge (R-\mu)R^{n-1}|a_n|,$$

and $|\arg a_j-\beta|\le\alpha\le\dfrac{\pi}{2}$, $1\le j\le n$ for some real α and β. Then for $0<\delta<1$ the number of zeros of P(z) in $\dfrac{|a_0|}{M'}\le|z|\le\delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where M and M' are as in Theorem 1 and Theorem B.

Similarly,combining Theorem 2 and Theorem C and then Theorem 3 and Theorem D we get the following results:

Corollary 5: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n where $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$ for $0\le j\le n$. Suppose that for some R>0, $0\le\mu<1, 0<\rho\le1, 0\le k\le n,$

$$0\ne\rho\alpha_0\le R\alpha_1\le R^2\alpha_2\le\dots\dots\dots\le R^{k-1}\alpha_{k-1}\le R^k\alpha_k$$
$$\ge R^{k+1}\alpha_{k+1}\ge\dots\dots\ge R^{n-1}\alpha_{n+1}\ge(R-\mu)R^{n-1}\alpha_n.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $\dfrac{|a_0|R}{M'}\le|z|\le\delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where M and M' are as in Theorem 2 and Theorem C.

Corollary 6: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n where

$\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$ for $0\le j\le n$. Suppose that for some

R>0, $0\le\mu<1, 0\le\lambda<1, 0<\rho_1\le1, 0<\rho_2\le1$ and for some $0\le k\le n, 0\le l\le n,$

$$0\ne\rho_1\alpha_0\le R\alpha_1\le R^2\alpha_2\le\dots\dots\dots\le R^{k-1}\alpha_{k-1}\le R^k\alpha_k$$
$$\ge R^{k+1}\alpha_{k+1}\ge\dots\dots\ge R^{n-1}\alpha_{n+1}\ge(R-\mu)R^{n-1}\alpha_n$$

and

$$\rho_2\beta_0\le R\beta_1\le R^2\beta_2\le\dots\dots\dots\le R^{l-1}\beta_{l-1}\le R^l\beta_l$$
$$\ge R^{l+1}\beta_{l+1}\ge\dots\dots\ge R^{n-1}\beta_{n+1}\ge(R-\lambda)R^{n-1}\beta_n.$$

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Then for $0<\delta<1$ the number of zeros of P(z) in the disc $\frac{|a_0|R}{M'}\le|z|\le\delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where M and M' are as in Theorem 3 and Theorem D.

For other different values of the parameters in the above results , we get many interesting results including generalizations of many known results in the literature.

III. LEMMAS

For the proof s of the above results, we make use of the following lemma which

Is due to Govil and Rahman [1]:

Lemma: For any two complex numbers z_1, z_2 such that $|z_1|\ge|z_2|$ and for some real α,β

, $|\arg z_j-\beta|\le\alpha\le\frac{\pi}{2}, j=1,2$, we have

$$|z_1-z_2|\le(|z_1|-|z_2|)\cos\alpha+(|z_1|+|z_2|)\sin\alpha\ .$$

IV. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

F(z)=(R-z)P(z)

$$=(R-z)(a_0+a_1z+a_2z^2+.....+a_{k-1}z^{k-1}+a_kz^k+a_{k+1}z^{k+1}+......+a_{n-1}z^{n-1}+a_nz^n)$$

$$=a_0R+(a_1R-a_0)z+(a_2R-a_1)z^2+......+(a_{k-1}R-a_{k-2})z^{k-1}+(a_kR-a_{k-1})z^k$$

$$+(a_{k+1}R-a_k)z^{k+1}+......+(a_{n-1}R-a_{n-2})z^{n-1}+(a_nR-a_{n-1})z^n-a_nz^{n+1}.$$

$$=a_0R+G(z)$$

where

$$G(z)=(a_1R-a_0)z+(a_2R-a_1)z^2+......+(a_{k-1}R-a_{k-2})z^{k-1}+(a_kR-a_{k-1})z^k$$

$$+(a_{k+1}R-a_k)z^{k+1}+......+(a_{n-1}R-a_{n-2})z^{n-1}+(a_nR-a_{n-1})z^n-a_nz^{n+1}.$$

For $|z|\le R$, we have, by using the hypothesis and the Lemma,

$$|G(z)|\le|a_1R-\rho a_0+\rho a_0-a_0|R+|a_2R-a_1|R^2+......+|a_{k-1}R-a_{k-2}|R^{k-1}+|a_kR-a_{k-1}|R^k$$

$$+|a_{k+1}R-a_k|R^{k+1}+......+|a_{n-1}R-a_{n-2}|R^{n-1}+|a_nR-\mu a_n+\mu a_n-a_{n-1}|R^n+|a_n|R^{n+1}$$

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$$\leq [(|a_1|R-\rho|a_0|)\cos\alpha+(|a_1|R+\rho|a_0|)\sin\alpha]R+(1-\rho)|a_0|R$$
$$+[(|a_2|R-|a_1|)\cos\alpha+(|a_2|R+|a_1|)\sin\alpha]R^2+......$$
$$+[(|a_{k-1}|R-|a_{k-2}|)\cos\alpha+(|a_{k-1}|R+|a_{k-2}|)\sin\alpha]R^{k-1}$$
$$+[(|a_k|R-|a_{k-1}|)\cos\alpha+(|a_k|R+|a_{k-1}|)\sin\alpha]R^k$$
$$+[(|a_k|-|a_{k+1}|R)\cos\alpha+(|a_k|+|a_{k+1}|R)\sin\alpha]R^{k-1}+......$$
$$+[(|a_{n-2}|-|a_{n-1}|R)\cos\alpha+(|a_{n-2}|+|a_{n-1}|R)\sin\alpha]R^{n-1}$$
$$+[\{|a_{n-1}|-(R-\mu)|a_n|\}\cos\alpha+\{|a_{n-1}|+(R-\mu)|a_n|\}\sin\alpha]R^n$$
$$+\mu|a_n|R^n+|a_n|R^{n+1}$$
$$=(1-\rho)|a_0|R-\rho|a_0|R\cos\alpha+\rho|a_0|R\sin\alpha+2|a_k|R^{k+1}\cos\alpha-|a_n|R^{n+1}\cos\alpha$$
$$+|a_n|R^{n+1}\sin\alpha+\mu|a_n|R^n\cos\alpha-\mu|a_n|R^n\sin\alpha+\mu|a_n|R^n+|a_n|R^{n+1}$$
$$+2\sin\alpha\sum_{j=1}^{n-1}|a_j|R^{j+1}$$
$$=|a_0|R-\rho|a_0|R-\rho|a_0|R\cos\alpha+\rho|a_0|R\sin\alpha+2|a_k|R^{k+1}\cos\alpha$$
$$-|a_n|R^{n+1}\cos\alpha+|a_n|R^{n+1}\sin\alpha+\mu|a_n|R^n\cos\alpha-\mu|a_n|R^n\sin\alpha+\mu|a_n|R^n+|a_n|R^{n+1}$$
$$+2\sin\alpha\sum_{j=1}^{n-1}|a_j|R^{j+1}$$
$$=|a_0|R-\rho|a_0|R(1+\cos\alpha-\sin\alpha)+2|a_k|R^{k+1}\cos\alpha-|a_n|R^{n+1}\cos\alpha$$
$$+|a_n|R^{n+1}\sin\alpha+\mu|a_n|R^n\cos\alpha-\mu|a_n|R^n\sin\alpha+\mu|a_n|R^n+|a_n|R^{n+1}$$
$$+2\sin\alpha\sum_{j=1}^{n-1}|a_j|R^{j+1}$$
$$=|a_0|R-\rho|a_0|R(1+\cos\alpha-\sin\alpha)+2|a_k|R^{k+1}\cos\alpha$$
$$+|a_n|R^{n+1}(1+\sin\alpha-\cos\alpha)+\mu|a_n|R^n(1+\cos\alpha-\sin\alpha)$$
$$+2\sin\alpha\sum_{j=0}^{n-1}|a_j|R^{j+1}$$
$$=M'.$$

Since F(z) is also analytic for $|z|\leq R$, F(0)=0, it follows by Schwarz lemma that

$$|G(z)|\leq M'|z| \quad \text{for } |z|\leq R.$$

Hence for $|z|\leq R$,

$$|F(z)|=|a_0R+G(z)|$$

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$$\geq |a_0|R - |G(z)|$$
$$\geq |a_0|R - M'|z|$$
$$> 0$$

if

$$|z| < \frac{|a_0|R}{M'}.$$

This shows that F(z) and hence P(z) has no zero in $|z| < \frac{|a_0|R}{M'}$, thereby proving Theorem 1.

Proof of Theorem 2: Consider the polynomial

F(z)=(R-z)P(z)

$$=(R-z)(a_0 + a_1 z + a_2 z^2 + ...+ a_{k-1}z^{k-1} + a_k z^k + a_{k+1}z^{k+1} + ...+ a_{n-1}z^{n-1} + a_n z^n)$$

$$= a_0 R + (a_1 R - a_0)z + (a_2 R - a_1)z^2 + ...+ (a_{k-1}R - a_{k-2})z^{k-1} + (a_k R - a_{k-1})z^k$$
$$+ (a_{k+1}R - a_k)z^{k+1} + ...+ (a_{n-1}R - a_{n-2})z^{n-1} + (a_n R - a_{n-1})z^n - a_n z^{n+1}$$

$$= a_0 R + (\alpha_1 R - \alpha_0)z + (\alpha_2 R - \alpha_1)z^2 + ...+ (\alpha_{k-1}R - \alpha_{k-2})z^{k-1} + (\alpha_k R - \alpha_{k-1})z^k$$
$$+ (\alpha_{k+1}R - \alpha_k)z^{k+1} + ...+ (\alpha_{n-1}R - \alpha_{n-2})z^{n-1} + (\alpha_n R - \alpha_{n-1})z^n$$
$$- \alpha_n z^{n+1} + i\sum_{j=1}^{n}(\beta_j R - \beta_{j-1}) - i\beta_n z^{n+1}$$

$$= a_0 R + G(z)$$

where

$$G(z) = (\alpha_1 R - \alpha_0)z + (\alpha_2 R - \alpha_1)z^2 + ...+ (\alpha_{k-1}R - \alpha_{k-2})z^{k-1} + (\alpha_k R - \alpha_{k-1})z^k$$
$$+ (\alpha_{k+1}R - \alpha_k)z^{k+1} + ...+ (\alpha_{n-1}R - \alpha_{n-2})z^{n-1} + (\alpha_n R - \alpha_{n-1})z^n$$
$$- \alpha_n z^{n+1} + i\sum_{j=1}^{n}(\beta_j R - \beta_{j-1})z^j - i\beta_n z^{n+1}$$

For $|z| \leq R$, we have, by using the hypothesis

$$|G(z)| \leq |\alpha_1 R - \rho\alpha_0 + \rho\alpha_0 - \alpha_0|R + |\alpha_2 R - \alpha_1|R^2 + ...+ |\alpha_{k-1}R - \alpha_{k-2}|R^{k-1} + |\alpha_k R - \alpha_{k-1}|R^k$$
$$+ |\alpha_{k+1}R - \alpha_k|R^{k+1} + ...+ |\alpha_{n-1}R - \alpha_{n-2}|)R^{n-1} + |\alpha_n R - \mu\alpha_n + \mu\alpha_n - \alpha_{n-1}|R^n$$
$$+ |\alpha_n|R^{n+1} + \sum_{j=1}^{n}|\beta_j R - \beta_{j-1}|R^j + |\beta_n|R^{n+1}$$

$$\leq (\alpha_1 R - \rho\alpha_0)R + (1-\rho)|\alpha_0|R + (\alpha_2 R - \alpha_1)R^2 + ...+ (\alpha_{k-1}R - \alpha_{k-2})R^{k-1}$$

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$$+(\alpha_k R-\alpha_{k-1})R^k+(\alpha_k-\alpha_{k+1}R)R^{k+1}+......+(\alpha_{n-2}-\alpha_{n-1}R)R^{n-1}$$

$$+[\{\alpha_{n-1}-(R-\mu)\alpha_n\}]R^n+\mu|\alpha_n|R^n+|\alpha_n|R^{n+1}+\sum_{j=1}^{n}(|\beta_j|R+|\beta_{j-1}|)R^j+|\beta_n|R^{n+1}$$

$$=|\alpha_0|R-\rho R(|\alpha_0|+\alpha_0)+2\alpha_k R^{k+1}+(|\alpha_n|-\alpha_n)R^{n+1}+\mu R^n(|\alpha_n|+\alpha_n)$$

$$+2\sum_{j=0}^{n}|\beta_j|R^{j+1}$$

$$=M'$$

Since F(z) is also analytic for $|z|\le R,$ F(0)=0, it follows by Schwarz lemma that

$$|G(z)|\le M'|z| \quad \text{for } |z|\le R.$$

Hence for $|z|\le R,$

$$|F(z)|=|a_0R+G(z)|$$

$$\ge|a_0|R-|G(z)|$$

$$\ge|a_0|R-M'|z|$$

$$>0$$

if

$$|z|<\frac{|a_0|R}{M'}.$$

This shows that F(z) and hence P(z) has no zero in $|z|<\dfrac{|a_0|R}{M'}$.

That completes the proof of Theorem 2.

Proof of Theorem 3: Consider the polynomial

F(z)=(R-z)P(z)

$$=(R-z)(a_0+a_1z+a_2z^2+.....+a_{k-1}z^{k-1}+a_kz^k+a_{k+1}z^{k+1}+......+a_{n-1}z^{n-1}+a_nz^n)$$

$$=a_0R+(a_1R-a_0)z+(a_2R-a_1)z^2+......+(a_{k-1}R-a_{k-2})z^{k-1}+(a_kR-a_{k-1})z^k$$

$$+(a_{k+1}R-a_k)z^{k+1}+......+(a_{n-1}R-a_{n-2})z^{n-1}+(a_nR-a_{n-1})z^n-a_nz^{n+1}$$

$$=a_0R+(\alpha_1R-\alpha_0)z+(\alpha_2R-\alpha_1)z^2+......+(\alpha_{k-1}R-\alpha_{k-2})z^{k-1}+(\alpha_kR-\alpha_{k-1})z^k$$

$$+(\alpha_{k+1}R-\alpha_k)z^{k+1}+......+(\alpha_{n-1}R-\alpha_{n-2})z^{n-1}+(\alpha_nR-\alpha_{n-1})z^n$$
$$-\alpha_nz^{n+1}+i[(\beta_1R-\beta_0)z+(\beta_2R-\beta_1)z^2+......+(\beta_{k-1}R-\beta_{k-2})z^{k-1}$$
$$+(\beta_kR-\beta_{k-1})z^k+(\beta_{k+1}R-\beta_k)z^{k+1}+......+(\beta_{n-1}R-\beta_{n-2})z^{n-1}$$
$$+(\beta_nR-\beta_{n-1})z^n-\beta_nz^{n+1}]$$

$$=a_0R+G(z)$$

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where

$$G(z) = (\alpha_1 R-\alpha_0)z+(\alpha_2 R-\alpha_1)z^2+......+(\alpha_{k-1}R-\alpha_{k-2})z^{k-1}+(\alpha_k R-\alpha_{k-1})z^k$$
$$+(\alpha_{k+1}R-\alpha_k)z^{k+1}+......+(\alpha_{n-1}R-\alpha_{n-2})z^{n-1}+(\alpha_n R-\alpha_{n-1})z^n$$
$$-\alpha_n z^{n+1}+i[(\beta_1 R-\beta_0)z+(\beta_2 R-\beta_1)z^2+......+(\beta_{l-1}R-\beta_{k-2})z^{l-1}$$
$$+(\beta_l R-\beta_{l-1})z^l+(\beta_{l+1}R-\beta_l)z^{l+1}+......+(\beta_{n-1}R-\beta_{n-2})z^{n-1}$$
$$+(\beta_n R-\beta_{n-1})z^n-\beta_n z^{n+1}]$$

For $|z|\le R$, we have, by using the hypothesis

$$|G(z)|\le|\alpha_1 R-\rho_1\alpha_0+\rho_1\alpha_0-\alpha_0|R+|\alpha_2 R-\alpha_1|R^2+......+|\alpha_{k-1}R-\alpha_{k-2}|R^{k-1}+|\alpha_k R-\alpha_{k-1}|R^k$$
$$+|\alpha_{k+1}R-\alpha_k|R^{k+1}+......+|\alpha_{n-1}R-\alpha_{n-2}|)R^{n-1}+|\alpha_n R-\mu\alpha_n+\mu\alpha_n-\alpha_{n-1}|R^n$$
$$+|\alpha_n|R^{n+1}+|\beta_1 R-\rho_2\beta_0+\rho_2\beta_0-\beta_0|R+|\beta_2 R-\beta_1|R^2+.......$$
$$+|\beta_{l-1}R-\beta_{l-2}|R^{l-1}+|\beta_l R-\beta_{l-1}|R^l+......+|\beta_{n-1}R-\beta_{n-2}|)R^{n-1}$$
$$+|\beta_n R-\lambda\beta_n+\lambda\beta_n-\beta_{n-1}|R^n+|\beta_n|R^{n+1}$$

$$\le(\alpha_1 R-\rho_1\alpha_0)R+(1-\rho_1)|\alpha_0|R+(\alpha_2 R-\alpha_1)R^2+......+(\alpha_{k-1}R-\alpha_{k-2})R^{k-1}$$
$$+(\alpha_k R-\alpha_{k-1})R^k+(\alpha_k-\alpha_{k+1}R)R^{k+1}+......+(\alpha_{n-2}-\alpha_{n-1}R)R^{n-1}$$
$$+[\{\alpha_{n-1}-(R-\mu)\alpha_n\}]R^n+\mu|\alpha_n|R^n+|\alpha_n|R^{n+1}+(\beta_1 R-\rho_2\beta_0)R$$
$$+(1-\rho_2)|\beta_0|R+(\beta_2 R-\beta_1)R^2+......+(\beta_{l-1}R-\beta_{l-2})R^{l-1}$$
$$+(\beta_l R-\beta_{l-1})R^l+(\beta_l-\beta_{l+1}R)R^{l+1}+......+(\beta_{n-2}-\beta_{n-1}R)R^{n-1}$$
$$+[\{\beta_{n-1}-(R-\lambda)\beta_n\}]R^n+\lambda|\beta_n|R^n+|\beta_n|R^{n+1}$$

$$=|\alpha_0|R-\rho_1 R(|\alpha_0|+\alpha_0)+2\alpha_k R^{k+1}+(|\alpha_n|-\alpha_n)R^{n+1}+\mu R^n(|\alpha_n|+\alpha_n)$$
$$+|\beta_0|R-\rho_2 R(|\beta_0|+\beta_0)+2\beta_l R^{l+1}+(|\beta_n|-\beta_n)R^{n+1}+\lambda R^n(|\beta_n|+\beta_n)$$
$$=M'$$

Since F(z) is also analytic for $|z|\le R,$ F(0)=0, it follows by Schwarz lemma that

$$|G(z)|\le M'|z| \quad \text{for } |z|\le R.$$

Hence for $|z|\le R,$

$$|F(z)|=|a_0 R+G(z)|$$
$$\ge|a_0|R-|G(z)|$$
$$\ge|a_0|R-M'|z|$$
$$>0$$

if

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$$|z| < \frac{|a_0|R}{M'} .$$

This shows that F(z) and hence P(z) has no zero in $|z| < \frac{|a_0|R}{M'}$.

That completes the proof of Theorem 3.

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