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Groups of Symmetrical Motions

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ABSTRACT

The objective of this paper is to unify the area of group theory with the study of symmetry. Group Theory is the mathematical study of symmetry and explores the general ways of studying it in many distinct settings. Dihedral group is the group of symmetries of a regular polygon, which includes rotations and reflections. Dihedral group plays an important role in group theory.

Group Theory and Symmetry- The study of symmetry has undergone tremendous chaı ges in late 19th and earlier 20th centuries with the development of group theory, Group theory has found applications in geometry, graph theory, physics, chemistry, architecture, crystallography and countless other areas of modern science. There is hardly a disciple in which the study of symmetry, often with the tools provided by group theory, has not played an important role.

A set G is group with the binary operation $*$ s.t.

(a) **Closure** : G is closed under operation $*$ i. e. if $a , b \in G$ then $a * b \in G$

(b) **Identity** : for all $a \in G \exists e \in G$ s. t. $a * e = a = e * a$

(c) **Inverse** : for all $a \in G$ there is an inverse in G

i.e. for all $a \in G \exists a' \in G$ s. t. $a * a' = e = a' * a$

(d) The operation $*$ acts **associativity.**

i.e. for all $a, b, c \in G$ we have $a * (b * c) = (a * b) * c$

Some permutation groups can be constructed by using symmetrical motions of certain geometrical figures. A motion of a geometrical figure is said to be symmetrical if the figure looks like the same after the motion as before.

By **rotation** of a plane figure, we mean a motion of the figure about any point in the figure.

By **reflection** of a plane figure, we mean a motion of the figure about a line such that every point of the line is kept fixed and every point not on the line is carried into the mirror image point at equal distance access the line.

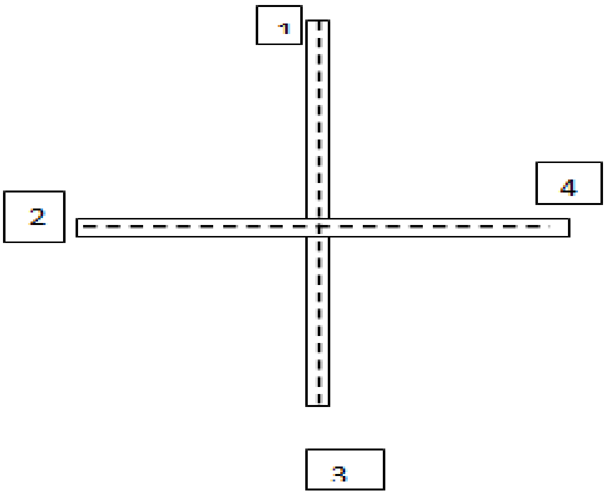
The resultant of two motions is a single motion arising from performing in succession the two motions.

Group of Symmetrical Rotations in the Plane:-

Consider the plane figure. Let $\rho_0, \rho_1, \rho_2, \rho_3$ denote the rotations counter clockwise about its centre O through the angles 0^o, 90^o, 180^o&270^o respectively, then the set G = $\{\rho_0, \rho_1, \rho_2, \rho_3\}$ is a group under the composition of resultant of motions.

Here ρ_0=i=ρ_1= (1 2 3 4), ρ_2= (1 3) (2 4) , ρ_3= (1 4 3 2)

*	• ρ_0	ρ_1	ρ_2	ρ_3
ρ_0	• ρ_0	ρ_1	ρ_2	ρ_3
ρ_1	• ρ_1	ρ_2	ρ_3	ρ_0
ρ_2	• ρ_2	ρ_3	ρ_0	ρ_1
ρ_3	• ρ_3	ρ_0	ρ_1	ρ_2



Here $(G,*)$ is a group. From the table, we see that composition of any two elements in G is also in G. Further, ρ_0 acts as an identity element. Each element in G also possesses the inverse. The pair of inverses are $(\rho_0, \rho_0), (\rho_1, \rho_3), (\rho_2, \rho_2), (\rho_3, \rho_1)$. The composition of elements is again associative as well i.e. for all $a, b, c \in G$ we have $a*(b*c) = (a*b)*c$

Group of symmetries of equilateral triangle:-

Consider an equilateral triangle whose vertices are labelled points. Consider a fixed point in the centre of this triangle. There are two types of symmetries we an look at. The first is counter clockwise rotational symmetries .We can rotate this triangle by 0^0 or equivalently 360^0 ,120^0 or 240^0 . Let ρ_0, ρ_1, ρ_2 denote these rotations respectively.

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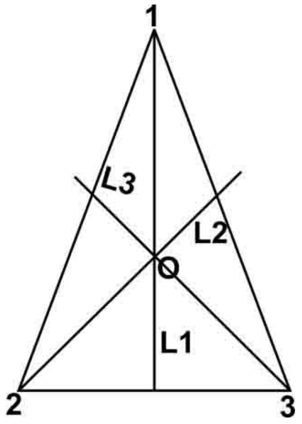

For each of these functions,

ρ_i :{1, 2, 3} -: {1, 2, 3} for i=0,1,2, we will use permutations of elements in a set as functions for which we have $\rho_i = \begin{pmatrix} 1 & 2 & 3 \\ \rho_i(1) & \rho_i(2) & \rho_i(3) \end{pmatrix}$ where the first row denotes the elements in f and the second row describes the images.

Thus, $\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

The second type of symmetries are mirror rotations about medians L_1, L_2, L_3. Let ν_1, ν_2, ν_3 these reflections respectively. Mirroring the equilateral triangle around each of these axes produces a symmetry.

$\nu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\nu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $\nu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

Then the set D_3= {$\rho_0, \rho_1, \rho_2, \nu_1, \nu_2, \nu_3$} is a non abelian group under the composition of resultant of motions. The composition table of $D_3$is shown below

O	ρ_0	ρ_1	ρ_2	ν_1	ν_2	ν_3
ρ_0	ρ_0	ρ_1	ρ_2	ν_1	ν_2	ν_3
ρ_1	ρ_1	ρ_2	ρ_0	ν_3	ν_1	ν_2
ρ_2	ρ_2	ρ_0	ρ_1	ν_2	ν_3	ν_1
ν_1	ν_1	ν_3	ν_2	ρ_0	ρ_2	ρ_1
ν_2	ν_2	ν_1	ν_3	ρ_1	ρ_0	ρ_2
ν_3	ν_3	ν_2	ν_1	ρ_2	ρ_1	ρ_0

We make note of following points

1. Every symmetry appears once in each row and in each column.

2. If f, g, h are symmetries of our triangle then it is clear that $(fog)oh = fo(goh)$

3. ρ_0 acts as an identity symmetry.

4. Every symmetry has an opposite or inverse symmetry. The pair of inverses are (ρ_0, ρ_0), (ρ_1, ρ_2), (ρ_2, ρ_1), (ν_1, ν_1), (ν_2, ν_2), (ν_3, ν_3).

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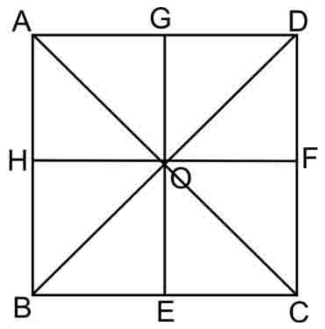
5 The symmetries are non commutative.

The group $D_3 = S_3$ is known as group of symmetries of an equilateral triangle or third dihedral group .

Group of Symmetries of Square :-



Consider a square and label the vertices as A, B, C, D.

Consider the first type of symmetry .Let the four rotations

about the centre O through 0^0 , 90^0 ,180^0 or 270^0

counter clockwise be denoted by $\rho_0, \ \rho_1, \rho_2, \rho_3$ respectively

Another type of symmetry is two reflections ν_1 and $\nu_2$about the diagonals AC and E
represent diagonal bisectors axial symmetries. The last type of symmetries are two reflecti
the perpendicular bisectors EG and HF . The set $D_4 = \{\rho_0, \ \rho_1, \rho_2, \rho_3, \ \nu_1, \nu_2, \ \nu_3, \nu_4\}$ is a group under the composition of resultant of motions, then

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},$$

$$\nu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \nu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}, \nu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \nu_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

Here, the set {1,2,3,4} refers to {A,B,C,D}.

The composition table of D_4 is shown as follows:

0	ρ_0	ρ_1	ρ_2	ρ_3	ν_1	ν_2	ν_3	ν_4
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	ν_1	ν_2	ν_3	ν_4
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	ν_4	ν_3	ν_1	ν_2
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	ν_2	ν_1	ν_4	ν_3
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	ν_3	ν_4	ν_2	ν_1
ν_1	ν_1	ν_3	ν_2	ν_4	ρ_0	ρ_2	ρ_1	ρ_3
ν_2	ν_2	ν_4	ν_1	ν_3	ρ_2	ρ_0	ρ_3	ρ_1
ν_3	ν_3	ν_2	ν_4	ν_1	ρ_3	ρ_1	ρ_0	ρ_2
ν_4	ν_4	ν_1	ν_3	ν_2	ρ_1	ρ_3	ρ_2	ρ_0

Every symmetry appears once in each row and in each column .

2 Associative law holds.

3 ρ_0 acts as an identity symmetry.

4 Every symmetry possesses an inverse symmetry. The pair of inverses are
$(\rho_0, \rho_0), (\rho_1, \rho_3), (\rho_2, \rho_2), (\rho_3, \rho_1), (\nu_1, \nu_1), (\nu_2, \nu_2), (\nu_3, \nu_3)$ and (ν_4, ν_4)

5. As seen from the composition table, we find that the symmetries are non abelian.

The group D_4 is called group of symmetries or fourth dihedral group or octic group.

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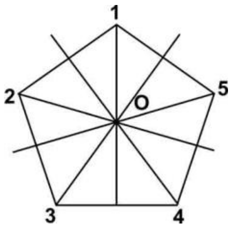
Group of symmetries of Pentagon:-

Consider a regular pentagon whose vertices are the labelled points.



There are rotation symmetries which we achieve by rotating the pentagon at angles in multiples of 360/n degrees. Let $\rho_0,\ \rho_1,\ \rho_2,\ \rho_3,\ \rho_4$ denote the rotations of pentagon by 0°, 72°, 144°, 216°, 288° respectively.

Thus $\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix},$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}, \rho_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix},$$

The other type of symmetries are given by reflection symmetries or axial flips. Axial flips are given by five axes of symmetries . Let $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5$ denote the reflections along the five axes of symmetry which pass through the centre.

Thus $\nu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}, \nu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix},$

$$\nu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}, \nu_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}, \nu_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix},$$

Cayley Table for D_5 is

0	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ν_1	ν_2	ν_3	ν_4	ν_5
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ν_1	ν_2	ν_3	ν_4	ν_5
ρ_1	ρ_1	ρ_2	ρ_3	ρ_4	ρ_0	ν_4	ν_5	ν_1	ν_2	ν_3
ρ_2	ρ_2	ρ_3	ρ_4	ρ_0	ρ_1	ν_2	ν_3	ν_4	ν_5	ν_1
ρ_3	ρ_3	ρ_4	ρ_0	ρ_1	ρ_2	ν_5	ν_1	ν_2	ν_3	ν_4
ρ_4	ρ_4	ρ_0	ρ_1	ρ_2	ρ_3	ν_3	ν_4	ν_5	ν_1	ν_2
ν_1	ν_1	ν_3	ν_5	ν_2	ν_4	ρ_0	ρ_3	ρ_1	ρ_4	ρ_2
ν_2	ν_2	ν_4	ν_1	ν_3	ν_5	ρ_2	ρ_0	ρ_3	ρ_1	ρ_4
ν_3	ν_3	ν_5	ν_2	ν_4	ν_1	ρ_4	ρ_2	ρ_0	ρ_3	ρ_1
ν_4	ν_4	ν_1	ν_3	ν_5	ν_2	ρ_1	ρ_4	ρ_2	ρ_0	ρ_3
ν_5	ν_5	ν_2	ν_4	ν_1	ν_3	ρ_3	ρ_1	ρ_4	ρ_2	ρ_0

From the above composition table, we see that

$D_5 = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}$ is non- abelian group.

1. ρ_0 acts as an identity element.
2. Closure property holds as the composition between any two symmetries of D_5 gives a symmetry in D_5.
3. Associative law holds between any three compositions.

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4. Each element in D_5 possesses an inverse in D_5 the fair of inverses are

$(\rho_0,\rho_0),(\rho_1,\rho_4),(\rho_2,\rho_3),(\rho_3,\rho_2),(\rho_4,\rho_1),(\nu_1,\nu_1),\ (\nu_2,\nu_2),(\nu_3,\nu_3),(\nu_4,\nu_4),(\nu_5,\nu_5),$

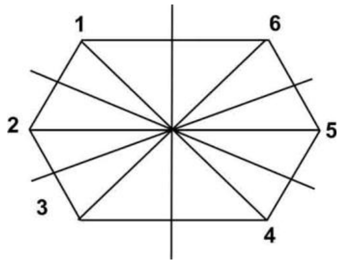
5. Commutative law does not hold as

$\rho_4 o \nu_2 = \nu_4 \qquad \nu_2 o \rho_4 = \nu_5$

$\rho_4\ o\,\nu_2 \neq \nu_2 o \rho_4$

Group of Symmetries of Hexagon :-

Consider a regular hexagon whose vertices are the labelled points.



There are six types o ... onsider by rotating the hexagon by angles in

multiples of 360/n de ... 0°, 180°, 240°, 300°.

Let ρ_0, ρ_1, ρ_2, ρ_3, ρ ... ly.

Thus, we have

$$\rho_0 = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 1 & 2 & 3 & 4 & 5 & 6\end{pmatrix}, \rho_1 = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 6 & 1 & 2 & 3 & 4 & 5\end{pmatrix},$$

$$\rho_2 = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 5 & 6 & 1 & 2 & 3 & 4\end{pmatrix}, \rho_3 = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 4 & 5 & 6 & 1 & 2 & 3\end{pmatrix},$$

$$\rho_4 = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 3 & 4 & 5 & 6 & 1 & 2\end{pmatrix}, \rho_5 = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 2 & 3 & 4 & 5 & 6 & 1\end{pmatrix},$$

The other type of symmetries are reflection symmetries or axial flips which are given along six axes of symmetry

ν_1:-(1) (35)(2 6)(4) $\qquad$ ν_2:- (3)(15)(2 4)(6)

ν_3:- (1 3)(4 6)(2)(5) $\qquad$ ν_4:- (16)(25) (3 4)

ν_5:- (1 2)(3 6) (4 5) $\qquad$ ν_6:- (14)(2 3) (5 6)

Composition table is

0	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6
ρ_1	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_0	ν_5	ν_4	ν_6	ν_1	ν_3	ν_2
ρ_2	ρ_2	ρ_3	ρ_4	ρ_5	ρ_0	ρ_1	ν_3	ν_1	ν_2	ν_5	ν_6	ν_4
ρ_3	ρ_3	ρ_4	ρ_5	ρ_0	ρ_1	ρ_2	ν_6	ν_5	ν_4	ν_3	ν_2	ν_1
ρ_4	ρ_4	ρ_5	ρ_0	ρ_1	ρ_2	ρ_3	ν_2	ν_3	ν_1	ν_6	ν_4	ν_5
ρ_5	ρ_5	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ν_4	ν_6	ν_5	ν_2	ν_1	ν_3
ν_1	ν_1	ν_4	ν_2	ν_6	ν_3	ν_5	ρ_0	ρ_2	ρ_4	ρ_1	ρ_5	ρ_3
ν_2	ν_2	ν_6	ν_3	ν_5	ν_1	ν_4	ρ_4	ρ_0	ρ_2	ρ_5	ρ_3	ρ_1
ν_3	ν_3	ν_5	ν_1	ν_4	ν_2	ν_6	ρ_2	ρ_4	ρ_0	ρ_3	ρ_1	ρ_5
ν_4	ν_4	ν_2	ν_6	ν_3	ν_5	ν_1	ρ_5	ρ_1	ρ_3	ρ_0	ρ_4	ρ_2
ν_5	ν_5	ν_1	ν_4	ν_2	ν_6	ν_3	ρ_1	ρ_3	ρ_5	ρ_2	ρ_0	ρ_4

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ν_6	ν_6	ν_3	ν_5	ν_1	ν_4	ν_2	ρ_3	ρ_5	ρ_1	ρ_4	ρ_2	ρ_0

From the composition table, it is clear that the composition of two symmetries from the set

$D_6 = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6\}$

is again in the set D_6.

ii) Associative law holds between any symmetries from the set D_6.

iii) ρ_0 acts as an identity symmetry.

iv) Each element in D_6 possesses an inverse in D_6. The pair of inverses are

$(\rho_0, \rho_0), (\rho_1, \rho_5), (\rho_2, \rho_4), (\rho_3, \rho_3), (\rho_4, \rho_2), (\rho_5, \rho_1),$

$(\nu_1, \nu_1), (\nu_2, \nu_2), (\nu_3, \nu_3) (\nu_4, \nu_4), (\nu_5, \nu_5), (\nu_6, \nu_6)$

The set D_6 above stated is non-abelian group and is called dihedral group D_6.

CONCLUSION

In general, we can say that dihedral group is the group of symmetries of the regular polygon which includes rotations and reflections. A regular polygon of n sides has exactly $2n$ different symmetries

1 n rotations about about the centre through the angles $0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots \frac{2(n-1)\pi}{n}$

2 n reflections about the lines joining the centre to n vertices (if n is odd) and $\frac{n}{2}$ reflections about the lines through the centre and parallel to the pair of parallel sides and $\frac{n}{2}$ about the lines through the centre and passing through the mid points to the pair of parallel sides (if n is even)

These $2n$ symmetries form a group under the composition of resultant of motions. This group is known as nth dihedral group and is denoted by D_n.

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