International Journal of Advance Research in Science and Engineering
Vol. No.6, Issue No. 09 , September 2017
www.ijarse.com
IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

Indexed Riesz Summability of an Infinite Series

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ABSTRACT

Certain results on indexed Cesaro summability were studied by Flett and Seyhan. Later similar results on absolute indexed Norlund summability were introduced by Misra. Extending their results, in this chapter a theorem on absolute indexed Riesz summability with additional parameter has been established.

Keywords: $|C,1|_k -$ ***Summability,*** $\phi - |C,1|_k -$ ***summability,*** $|N, p_n|_k$ - ***summability,*** $\varphi - |\overline{N}, p_n|_k, k \geq 1$**, *summability,*** $\varphi - |\overline{N}, p_n, \delta, \gamma|_k$ **- *summability***

Ams Classification No: 40d25

I. INTRODUCTION

Let $\{s_n\}$ denote the nth partial sum of an infinite series $\sum a_n$ and let $\{p_n\}$ be a sequence of positive real constants such that

(1.1) $$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \text{ for } n = 0,1,2,... \left(P_i = p_i = 0,\ i < 0\right).$$

Then the sequence-to-sequence transformation given by

(1.2) $$T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu \ ,$$

defines the $(\overline{N}, p_n)$ mean of the sequence $\{s_n\}$.

The series $\sum a_n$ is said to be $|\overline{N}, p_n|_k, k \geq 1$ summable [1], if

(1.3) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty.,$$

Taking p_n=1 for all n, $|\overline{N}, p_n|_k$ summability reduces to $|C,1|_k$ summability method.

Further, for a sequence $\{\phi_n\}$ of positive real numbers the series $\sum a_n$ is said to be $\phi - |\overline{N}, p_n|_k, k \geq 1$, summable if

(1.4) $$\sum_{n=1}^{\infty} \varphi_n^{k-1} |T_n - T_{n-1}|^k < \infty..$$

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

Taking $\varphi_n = \dfrac{P_n}{p_n}$, for all n, $\phi - \left|\overline{N}, p_n\right|_k$ – Summability method reduces to $\left|\overline{N}, p_n\right|_k$ – Summability method. The series $\sum a_n$ is said to be $\varphi - \left|\overline{N}, p_n.\delta\right|_k, k \geq 1, \delta \geq 0$, summable if

(1.5) $$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \left|T_n - T_{n-1}\right|^k < \infty.$$

Taking $\delta = 0$, $\phi - \left|\overline{N}, p_n.\delta\right|_k$ – Summability method reduces to $\phi - \left|\overline{N}, p_n\right|_k$ – Summability method.

For any real number γ the series $\sum a_n$ is said to be $\phi - \left|\overline{N}, p_n, \delta, \gamma\right|_k, k \geq 1, \delta \geq 0$, summable if

(1.6) $$\sum_{n=1}^{\infty} \phi_n^{\gamma(\delta k+k-1)} \left|T_n - T_{n-1}\right|^k < \infty.$$

II. KNOWN THEOREMS

Concerning with $\left|C,1\right|_k$ – Summability of infinite series $\sum a_n$, in 1957, Flett [2] has established the following result. He proved the following theorem.

2.1. Theorem

Let σ_n and τ_n denote the $(C,1)$ mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively that is

(i) $$\sigma_n = \frac{1}{n+1}\sum_{\nu=0}^{n} s_\nu$$

and

(ii) $$\tau_n = \frac{1}{n+1}\sum_{\nu=0}^{n} \nu a_\nu .$$

Then the series $\sum a_n$ is summable $\left|C,1\right|_k, k \geq 1$, if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n}\left|\tau_n\right|^k < \infty.$$

Further in 1995, Seyhan [6] extended the result of Flett to $\phi - \left|C,1\right|_k$ – summability by establishing

2.2. Theorem:

Let σ_n and τ_n be as defined in theorem-A and let $\{\varphi_n\}$ be a sequence of positive real numbers. Then the series $\sum a_n$ is summable $\varphi - \left|C,1\right|_k, k \geq 1$, if and only if

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k}\left|\tau_n\right|^k < \infty.$$

In 2010, Mishra et al. [3] have established a theorem similar to theorem-A, by $\left|\overline{N}, p_n\right|_k$ – summability method. They proved:

2.3. Theorem

Let $\{t_n\}$ denote the $(\overline{N}, p_n)$-mean of the sequence $\{na_n\}$ and $\{T_n\}$ be the sequence as defined in (5.1.2), where $\{p_n\}$ be a sequence of positive real constants satisfying the following conditions:

(a) $np_n = O(P_n),$

(b) $P_n = O(np_n),$

and

(c) $n|\Delta p_n| = O(p_n),$

then $\sum a_n$ is summable $\left|\overline{N}, p_n\right|_k, k \geq 1$, if and only if $\sum_{n=1}^{\infty} \frac{|t_n|^k}{n} < \infty.$

Recently Misra et al [4] established a similar theorem for $\varphi - \left|\overline{N}, p_n\right|_k, k \geq 1$, summability method. They proveed the following:

2.4. Theorem

Let $\{T_n\}$ and $\{t_n\}$ denote the sequences of $(\overline{N}, p_n)$-mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively. Let $\{\varphi_n\}$, $\{p_n\}$ be the sequences of positive real constants satisfying the following conditions:

(i) $np_n = O(P_n)$ (ii) $P_n = O(np_n)$ (iii) $n|\Delta p_n| = O(p_n)$ and (iv) $\frac{\phi_n}{n} = O(1)$.

Then the series $\sum a_n$ is summable $\varphi - \left|\overline{N}, p_n\right|_k, k \geq 1$, if and only if

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n|^k < \infty.$$

\Subsequently, extending theore-2.4, Misra et al [5] established the following theorem:

2.5. Theorem

Let $\{T_n\}$ and $\{t_n\}$ denote the sequences of $(\overline{N}, p_n)$ - mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively. Let $\{\varphi_n\}$, $\{p_n\}$ be the sequences of positive real constants satisfying the following conditions:

(i) $np_n = O(P_n)$ (ii) $P_n = O(np_n)$ (iii) $n|\Delta p_n| = O(p_n)$ and (iv) $\frac{\phi_n}{n} = O(1)$.

Then the series $\sum a_n$ is summable $\phi - \left|\overline{N}, p_n, \delta\right|_k, k \geq 1, \delta \geq 0$, if and only if

$$\sum_{n=1}^{\infty} \frac{\phi_n^{\delta k+k-1}}{n^k} |t_n|^k < \infty.$$

In what follows, in this paper we prove a similar theorem on $\phi-\left|\bar{N}, p_n, \delta, \gamma\right|_k, k \geq 1, \delta \geq 0,$ summabilty method. We prove:

III. MAIN THEOREM

Let $\{\varphi_n\}$, $\{p_n\}$ be the sequences of positive real constants such that

(3.1) $$np_n = O(P_n)$$

(3.2) $$P_n = O(np_n)$$

(3.3) $$n|\Delta p_n| = O(p_n)$$

(3.4) $\left\{\phi_n^{\gamma(\delta k+k-1)}\right\}, k \geq 1, \delta \geq 0,$ is monotonically decreasing.

Let $\{T_n\}$ and $\{t_n\}$ denote the sequences of $(\bar{N}, p_n)$-mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively. Then for any real number γ the series $\sum a_n$ is summable $\phi-\left|\bar{N}, p_n, \delta, \gamma\right|_k, k \geq 1, \delta \geq 0$, if and only if

(3.5) $$\sum_{n=1}^{\infty} \frac{\phi_n^{\gamma(\delta k+k-1)}}{n^k} |t_n|^k < \infty.$$

IV. REQUIRED LEMMA

We require the following lemma to prove our theorem.

4.1. Lemma [3]:

Let $\{p_n\}$ be a sequence of positive real constants satisfying (i) and (ii) of Theorem-C, then

(4.1.1) $$p_{n+1} = O(p_n)$$

and

(4.1.2) $$p_n = O(p_{n+1})$$

hold good.

V. PROOF OF THE MAIN THEOREM

Sufficient Part $(\Leftarrow)$:

Since $\{t_n\}$ is the $(\bar{N}, p_n)$-mean of the sequence $\{na_n\}$, we have

(5.1) $$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \nu a_\nu = \frac{1}{P_n}\sum_{\nu=1}^{n} p_\nu \nu a_\nu$$

Then $$P_n t_n - P_{n-1}t_{n-1} = np_n a_n$$

(5.2) $$\Rightarrow a_n = \frac{P_n t_n - P_{n-1}t_{n-1}}{np_n}$$

Now, we have

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{\lambda=0}^{\nu} a_\lambda$$
$$= \frac{1}{P_n}\sum_{\lambda=0}^{n} a_\lambda \sum_{\nu=\lambda}^{n} p_\nu = \frac{1}{P_n}\sum_{\lambda=0}^{n} a_\lambda (P_n - P_{\lambda-1})$$
$$= \sum_{\lambda=0}^{n} a_\lambda - \frac{1}{P_n}\sum_{\lambda=1}^{n} a_\lambda P_{\lambda-1}.$$

Then,

(5.3)
$$\nabla T_n = T_n - T_{n-1}$$
$$= \sum_{\lambda=0}^{n} a_\lambda - \frac{1}{P_n}\sum_{\lambda=1}^{n} a_\lambda P_{\lambda-1} - \sum_{\lambda=0}^{n-1} a_\lambda + \frac{1}{P_{n-1}}\sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1}$$
$$= a_n + \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1} - \frac{P_{n-1}a_n}{P_n}$$
$$= a_n\left(1 - \frac{P_{n-1}}{P_n}\right) + \frac{p_n}{P_n P_{n-1}}\sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1}$$
$$= \frac{p_n}{P_n P_{n-1}}\sum_{\lambda=1}^{n} a_\lambda P_{\lambda-1}$$

Using (5.2), we get

$$\nabla T_n = \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}\frac{P_\nu t_\nu - P_{\nu-1}t_{\nu-1}}{\nu p_\nu}$$
$$= \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n}\frac{P_{\nu-1}P_\nu t_\nu}{\nu p_\nu} - \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n}\frac{P_{\nu-1}^2 t_{\nu-1}}{\nu p_\nu}$$
$$= \frac{t_n}{n} + \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1}\frac{P_{\nu-1}P_\nu t_\nu}{\nu p_\nu} - \frac{p_n}{P_n P_{n-1}}\sum_{\nu=0}^{n-1}\frac{P_\nu^2 t_\nu}{(\nu+1)p_{\nu+1}}$$
$$= \frac{t_n}{n} + \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1} P_\nu t_\nu\left(\frac{P_{\nu-1}}{\nu p_\nu} - \frac{P_\nu}{(\nu+1)p_{\nu+1}}\right)$$
$$= \frac{t_n}{n} - \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1}\frac{P_\nu t_\nu}{\nu} + \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1} P_\nu^2 t_\nu\left(\frac{1}{\nu p_\nu} - \frac{1}{(\nu+1)p_{\nu+1}}\right)$$
$$= \frac{t_n}{n} - \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1}\frac{P_\nu t_\nu}{\nu} + \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1}\frac{P_\nu^2 t_\nu}{\nu(\nu+1)p_{\nu+1}}$$
$$+ \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1}\frac{P_\nu^2 t_\nu}{\nu}\left(\frac{1}{p_\nu} - \frac{1}{p_{\nu+1}}\right)$$
$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ (Say)}.$$

In order to complete the proof of the sufficient part, by using Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\phi_n^{\gamma(\delta k+k-1)}|T_{n,i}|^k < \infty \text{ for } i = 1,2,3,4.$$

Now, we have

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

$$\sum_{n=1}^{\infty}\phi_n^{\gamma(\delta k+k-1)}\left|T_{n,1}\right|^k=\sum_{n=1}^{\infty}\frac{\phi_n^{\gamma(\delta k+k-1)}}{n^k}\left|t_n\right|^k<\infty.\text{ By (3.5).}$$

Next,

$$\sum_{n=2}^{m+1}\phi_n^{\gamma(\delta k+k-1)}\left|T_{n,2}\right|^k\le\sum_{n=2}^{m+1}\phi_n^{\gamma(\delta k+k-1)}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left(\sum_{\nu=1}^{n-1}\frac{P_\nu\left|t_\nu\right|}{\nu}\right)^k$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\left|t_\nu\right|p_\nu\right)^k\quad\text{, using (3.2)}$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}p_\nu\right)^{k-1}\left(\sum_{\nu=1}^{n-1}p_\nu\left|t_\nu\right|^k\right)$$

(Using Holder's inequality)

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\delta k+k-1}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{\nu=1}^{n-1}\left|t_\nu\right|^k p_\nu\right)$$

$$=O(1)\sum_{\nu=1}^{m}p_\nu\left|t_\nu\right|^k\sum_{n=\nu+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}$$

$$=O(1)\sum_{\nu=1}^{m}p_\nu\left|t_\nu\right|^k\left(\frac{\phi_\nu p_\nu}{P_\nu}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_\nu}{p_\nu}\right)^{\gamma(\delta k+k-1)-k+1}\sum_{n=\nu+1}^{m+1}\left(\frac{1}{P_{n-1}}-\frac{1}{P_n}\right)\quad\text{, using}$$

(3.4)

$$=O(1)\sum_{\nu=1}^{m}p_\nu\left|t_\nu\right|^k\left(\frac{\phi_\nu p_\nu}{P_\nu}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_\nu}{p_\nu}\right)^{\gamma(\delta k+k-1)-k+1}\frac{1}{P_\nu}$$

$$=O(1)\sum_{\nu=1}^{m}\left(\frac{\phi_\nu p_\nu}{P_\nu}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_\nu}{p_\nu}\right)^{\gamma(\delta k+k-1)-k}\left|t_\nu\right|^k$$

$$=O(1)\sum_{\nu=1}^{m}\frac{\phi_\nu^{\lambda(\delta k+k-1)}}{\nu^k}\left|t_\nu\right|^k\text{, using (3.2)}$$

$$=O(1)\text{, using (3.5)}$$

Furthermore,

$$\sum_{n=2}^{m+1}\phi_n^{\gamma(\delta k+k-1)}\left|T_{n,3}\right|^k\le\sum_{n=2}^{m+1}\phi_n^{\gamma(\delta k+k-1)}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left(\sum_{\nu=1}^{n-1}\frac{P_\nu^2\left|t_\nu\right|}{\nu(\nu+1)p_{\nu+1}}\right)^k$$

$$\le\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{P_\nu P_{\nu+1}\left|t_\nu\right|}{\nu(\nu+1)p_{\nu+1}}\right)^k$$

$$\le\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\left|t_\nu\right|p_\nu\right)^k\text{, using (3.2)}$$

$$=O(1)\ as\ m\to\infty\text{, proceeding as above.}$$

Next, we have

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

$$\sum_{n=2}^{m+1}\phi_n^{\gamma(\delta k+k-1)}\left|T_{n,4}\right|^k \le \sum_{n=2}^{m+1}\phi_n^{\gamma(\delta k+k-1)}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left(\sum_{\nu=1}^{n-1}\frac{P_\nu^2\left|t_\nu\right|}{\nu}\left|\frac{1}{p_\nu}-\frac{1}{p_{\nu+1}}\right|\right)^k$$

$$\le\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{P_\nu^2\left|t_\nu\right|}{\nu}\left|\frac{p_{\nu+1}-p_\nu}{p_{\nu+1}p_\nu}\right|\right)^k,$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\nu\left|\Delta p_\nu\right|\left|t_\nu\right|\right)^k, \text{ Using (3.2)}$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}p_\nu\left|t_\nu\right|\right)^k, \text{ Using (3.3)}$$

$$=O(1)\ as\ m\to\infty, \text{ proceeding as above.}$$

This proves the sufficient part of the theorem.

Necessary Part $(\Rightarrow)$:

From (5.3), we have $\dfrac{P_{n-1}P_n}{p_n}\nabla T_n=\sum_{\nu=1}^{n}P_{\nu-1}a_\nu$,

where

$$a_n=\frac{1}{P_{n-1}}\left[\frac{P_{n-1}P_n}{p_n}\nabla T_n-\frac{P_{n-2}P_{n-1}}{p_{n-1}}\nabla T_{n-1}\right]$$

$$=\frac{P_n}{p_n}\nabla T_n-\frac{P_{n-2}}{p_{n-1}}\nabla T_{n-1}$$

Now, we have

$$t_n=\frac{1}{P_n}\sum_{\nu=1}^{n}p_\nu\nu\,a_\nu$$

$$=\frac{1}{P_n}\sum_{\nu=1}^{n}\left(\nu P_\nu\nabla T_\nu-\frac{\nu\,p_\nu P_{\nu-2}}{p_{\nu-1}}\nabla T_{\nu-1}\right)$$

$$=\frac{1}{P_n}\sum_{\nu=1}^{n}\nu P_\nu\nabla T_\nu-\frac{1}{P_n}\sum_{\nu=0}^{n-1}\frac{(\nu+1)p_{\nu+1}P_{\nu-1}}{p_\nu}\nabla T_\nu$$

$$=n\nabla T_n+\frac{1}{P_n}\sum_{\nu=1}^{n}\nu P_\nu\nabla T_\nu-\frac{1}{P_n}\sum_{\nu=1}^{n-1}\frac{(\nu+1)p_{\nu+1}P_{\nu-1}}{p_\nu}\nabla T_\nu$$

$$=n\nabla T_n+\frac{1}{P_n}\sum_{\nu=1}^{n}\nu\nabla T_\nu\left(P_\nu-P_{\nu-1}\right)$$

$$+\frac{1}{P_n}\sum_{\nu=1}^{n-1}\nu\nabla T_\nu P_{\nu-1}\left(1-\frac{p_{\nu+1}}{p_\nu}\right)+\frac{1}{P_n}\sum_{\nu=1}^{n-1}P_{\nu-1}\frac{p_{\nu+1}}{p_\nu}\nabla T_\nu$$

$$=t_{n,1}+t_{n,2}+t_{n,3}+t_{n,4}, \text{ say.}$$

To complete the necessary part, using Minokowski's inequality, we need to show only

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

$$\sum_{n=1}^{\infty} \frac{\phi_n^{\gamma(\delta k+k-1)}}{n^k} \left| t_{n,i} \right|^k < \infty. \; for \; i = 1, 2, 3, 4.$$

We have,
$$\sum_{n=1}^{\infty} \frac{\phi_n^{\gamma(\delta k+k-1)}}{n^k} \left| t_{n,1} \right|^k = \sum_{n=1}^{\infty} \frac{\phi_n^{\gamma(\delta k+k-1)} n^k}{n^k} \left| \nabla T_n \right|^k$$
$$= \sum_{n=1}^{\infty} \phi_n^{\gamma(\delta k+k-1)} \left| \nabla T_n \right|^k$$
$$= O(1).$$

Further,
$$\sum_{n=1}^{\infty} \frac{\phi_n^{\gamma(\delta k+k-1)}}{n^k} \left| t_{n,2} \right|^k \le \sum_{n=2}^{m+1} \frac{\phi_n^{\gamma(\delta k+k-1)}}{n^k} \frac{1}{P_n^k} \left(\sum_{\nu=1}^{n-1} \nu \left| \nabla T_\nu \right| p_\nu \right)^k$$
$$\le \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\gamma(\delta k+k-1)} \left(\frac{P_n}{p_n} \right)^{\gamma(\delta k+k-1)-k+1} \frac{p_n}{P_n^k P_{n-1}} \left(\sum_{\nu=1}^{n-1} \nu \left| \nabla T_\nu \right| p_\nu \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\gamma(\delta k+k-1)} \left(\frac{P_n}{p_n} \right)^{\gamma(\delta k+k-1)-k+1} \frac{p_n}{P_n^k P_{n-1}} \left(\sum_{\nu=1}^{n-1} p_\nu \right)^{k-1} \left(\sum_{\nu=1}^{n-1} \nu^k \left| \nabla T_\nu \right|^k p_\nu \right),$$

Using Holder's inequality

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\gamma(\delta k+k-1)} \left(\frac{P_n}{p_n} \right)^{\gamma(\delta k+k-1)-k+1} \frac{p_n}{P_n P_{n-1}} \left(\sum_{\nu=1}^{n-1} \nu^k \left| \nabla T_\nu \right|^k p_\nu \right)$$

$$= O(1) \sum_{\nu=1}^{m} p_\nu \nu^k \left| \nabla T_\nu \right|^k \sum_{n=\nu+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \left(\frac{\phi_n p_n}{P_n} \right)^{\gamma(\delta k+k-1)} \left(\frac{P_n}{p_n} \right)^{\gamma(\delta k+k-1)-k+1}$$
$$= O(1) \sum_{\nu=1}^{m} p_\nu \nu^k \left| \nabla T_\nu \right|^k \sum_{n=\nu+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) (\phi_n)^{\gamma(\delta k+k-1)} \left(\frac{p_n}{P_n} \right)^{k-1}$$
$$= O(1) \sum_{\nu=1}^{m} p_\nu \nu^k \left| \nabla T_\nu \right|^k \left(\frac{1}{P_\nu} - \frac{1}{P_{m+1}} \right) (\phi_\nu)^{\gamma(\delta k+k-1)} \left(\frac{p_\nu}{P_\nu} \right)^{k-1}$$, using (3.4)
$$\le O(1) \sum_{\nu=1}^{m} \frac{p_\nu}{P_\nu} \nu^k \left| \nabla T_\nu \right|^k (\phi_\nu)^{\gamma(\delta k+k-1)} \left(\frac{p_\nu}{P_\nu} \right)^{k-1}$$
$$= O(1) \sum_{\nu=1}^{m} \phi_\nu^{\gamma(\delta k+k-1)} \left| \nabla T_\nu \right|^k$$
$$= O(1) \; as \; m \to \infty$$, using (3.5)

Again,
$$\sum_{n=1}^{\infty} \frac{\phi_n^{\gamma(\delta k+k-1)}}{n^k} \left| t_{n,3} \right|^k \le \sum_{n=2}^{m+1} \frac{\phi_n^{\gamma(\delta k+k-1)}}{n^k} \frac{1}{P_n^k} \left(\sum_{\nu=1}^{n-1} \nu P_{\nu-1} \left| \nabla T_\nu \right| \frac{\left| p_\nu - p_{\nu-1} \right|}{p_\nu} \right)^k$$

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n-1}\nu P_{\nu-1}|\nabla T_\nu|\frac{|\Delta p_\nu|}{p_\nu}\right)^k,$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n}P_{\nu-1}|\nabla T_\nu|\right)^k, \text{ using (3.3)}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{n^{k-1}}{P_n^k}\left(\sum_{\nu=1}^{n}p_\nu|\nabla T_\nu|\right)^k, \text{ using (3.2)}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{n^{k-1}}{P_n^k}\left(\sum_{\nu=1}^{n}p_\nu\right)^{k-1}\left(\sum_{\nu=1}^{n}p_\nu|\nabla T_\nu|^k\right)$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{n^{k-1}}{P_n}\left(\sum_{\nu=1}^{n}p_\nu|\nabla T_\nu|^k\right)$$

$$= O(1)\sum_{\nu=1}^{m}p_\nu|\nabla T_\nu|^k\sum_{n=\nu+1}^{m+1}(\phi_n)^{\gamma(\delta k+k-1)}\left(\frac{p_n}{P_n}\right)^{k-1}\frac{n^{k-1}}{P_n}$$

$$= O(1)\sum_{\nu=1}^{m}p_\nu|\nabla T_\nu|^k(\phi_\nu)^{\gamma(\delta k+k-1)}\frac{\nu}{P_\nu}, \text{ using (3.4)}$$

$$= O(1)\sum_{\nu=1}^{m}\phi_\nu^{\gamma(\delta k+k-1)}|\nabla T_\nu|^k$$

$$= O(1) \text{ as } m\to\infty, \text{ using (3.5).}$$

Finally,

$$\sum_{n=1}^{\infty}\frac{\phi_n^{\gamma(\delta k+k-1)}}{n^k}|t_{n,4}|^k \le \sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n-1}P_{\nu-1}\frac{p_{\nu+1}}{p_\nu}|\nabla T_\nu|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k+1}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n-1}P_{\nu-1}|\nabla T_\nu|\right)^k, \text{ by lemma 4.1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\phi_n p_n}{P_n}\right)^{\gamma(\delta k+k-1)}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n-1}P_\nu|\nabla T_\nu|\right)^k$$

$$= O(1) \text{ as } m\to\infty, \text{ proceeding as above.}$$

This completes the proof of the theorem.

VI. CONCLUSION

Our theorem generalizes theorem-2.4. In fact, by putting $\delta = 0$ in our theorem, theorem-2.4 is obtained as a particular case of our theorem.

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ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

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