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Approximation of signals belonging to generalized Lipschitz class using $(\overline{N}, p_n, q_n)(E, s)$-summability mean of conjugate series of Fourier series

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ABSTRACT

Degree of approximation of functions of different classes has been studied by several researchers by using different summability methods. In the proposed paper a new theorem has been established for the approximation of a signal (function) belonging to the $W(L_r, \xi(t))$-class by $(\overline{N}, p_n, q_n)(E, s)$-product summability means of conjugate series of a Fourier series. The result obtained here is a generalization of several known theorems.

Key Words: Degree of approximation; conjugate of the Fourier series; Weighted $W(L_r, \xi(t))$-class; $(\overline{N}, p_n, q_n)(E, s)$-mean; Lebesgue integral.

AMS Subject classification (2010): Primary 42A24; secondary 41A25, 42B025, 42B08, 40C05.

I. INTRODUCTION

Fourier series and operators are considered as representations of functions or signals and they are of incredible significance in both hypothetical and practical areas. For real and complex sequences working with infinite matrices the summability method is directly related to broad area of engineering and sciences. The approximation properties for the periodic signals in $L^r(r \geq 1)$-spaces, Lipschitz classes $Lip(\alpha)$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and weighted Lipschitz class $W(L_r, \xi(t))$ through a Fourier series is known as Fourier approximation. The engineer and scientist uses the properties of the Fourier approximation for designing digital filters. Especially Psarakis and Moustakides [9] presented a new L_2 based method for designing Finite Impulse Response digital filters for getting optimum approximation. In a similar manner, L_p-space, L_2-space and L_∞-space plays an important role for designing digital filters for getting optimum approximation. The approximation of signals belonging to various Lischitz classes through trigonometric Fourier series using different summability mean has been proved by various investigators like Lal [3], Mishra and Sonavane [4], Mishra *et al.* [5], Nigam and Sharma [7], Pradhan *et al.* [8] and many others. Subsequently, Nigam and Sharma [6] has proved a theorem on the approximation of functions belonging to $W(L_r, \xi(t))$ class by $(C, 1)(E, q)$ summability means of conjugate series of Fourier series. In an attempt to make an advance study in this direction, in this paper, we have proved a new theorem on the approximation of functions belonging to $W(L_r, \xi(t))$-class by $(\overline{N}, p_n, q_n)(E, s)$-summability mean of conjugate series of the Fourier series which generalizes several known results.

II. DEFINITIONS AND NOTATIONS

Let $\sum u_n$ be a given infinite series with the sequence of partial sum $\{s_n\}$. Let $\{p_n\}$ and $\{q_n\}$ be sequences of positive real numbers such that,

$$P_n = \sum_{k=0}^{n} p_k, \;\; Q_n = \sum_{k=0}^{n} q_k,$$

and

$$R_n = p_0 q_n + p_1 q_{n-1} + ... + p_n q_0 (\neq 0), \;\; p_{-1} = q_{-1} = R_{-1} = 0.$$

The sequence to sequence transformation,

$$t_n^{\overline{N}} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k s_k, \tag{2.1}$$

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IJARSE
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defines the $(\overline{N}, p_n, q_n)$ mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$ and $\{q_n\}$, and it is denoted by $t_n^{\overline{N}}$.

If $\lim_{n\to\infty} t_n^{\overline{N}} \to s$, then the series $\sum u_n$ is $(\overline{N}, p_n, q_n)$ summable to s.
The necessary and sufficient conditions for regularity of $(\overline{N}, p_n, q_n)$ summability are:

$$\text{(i)}\ \frac{p_k q_k}{R_n} \to 0, \text{ for each integer } k \geq 0, \text{ as } n \to \infty, \text{ and}$$

$$\text{(ii)}\ \left|\sum_{k=0}^{n} p_k q_k\right| < C|R_n|, \text{where, C is any positive integer independent of n.}$$

The sequence to sequence transformation,

$$E_n^s = \frac{1}{(1+s)^n} \sum_{v=0}^{n} \binom{n}{v} s^{n-v} s_v, \tag{2.2}$$

defines the (E, s) mean of the sequence $\{s_n\}$ and is denoted by E_n^s.
If $E_n^s \to s$ as $n \to \infty$, then $\sum u_n$ is summable to s with respect to (E, s) summability. Also (E, s) method is regular (see [1]).

Now, we define a new composite transformation $(\overline{N}, p_n, q_n)$ over (E, s) of $\{s_n\}$ as

$$T_n^{\overline{N}E} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k \left(E_k^s\right) = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k} \sum_{v=0}^{k} \binom{k}{v} s^{k-v} s_v\right\}. \tag{2.3}$$

If $T_n^{\overline{N}E} \to s$ as $n \to \infty$, then $\sum u_n$ is summable to s by $(\overline{N}, p_n, q_n)(E, s)$ summability method.
Further as $(\overline{N}, p_n, q_n)$ and (E, s) method are both regular, so $(\overline{N}, p_n, q_n)(E, s)$ method is also regular.
Thus, we may write,

$$T_n^{\overline{N}E} = t_n^{\overline{N}}(E_n^s(s_n)) \to s \quad (n \to \infty).$$

Remark 1. If we put $q_n = 1$ in equation (2.3), then $(\overline{N}, p_n, q_n)(E, s)$-summability method reduces to $(\overline{N}, p_n)(E, s)$-summability. Again if we put $p_n = 1$ and $q_n = 1$ in equation (2.3), it reduces to $(C, 1)(E, s)$-summability.

Let f be a 2π periodic function belonging to $L^r[0, 2\pi]$ $(r \geq 1)$, with the partial sum $s_n(f)$, then

$$s_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx). \tag{2.4}$$

The conjugate series of the Fourier series (2.4) is given by

$$\tilde{s_n}(f) = \sum_{k=1}^{\infty} (a_k \cos kx - b_k \sin kx). \tag{2.5}$$

Here, as regards to signals (functions) belonging to various Lipschitz classes we may recall that,

(a) $f \in Lip(\alpha)$, if

$$|f(x+t) - f(x)| = O(|t|^\alpha) \text{ for } 0 < \alpha \leq 1,\ t > 0;$$

(b) $f \in Lip(\alpha, r)$, for $0 \leq x \leq 2\pi$, if

$$\left(\int_{[0,2\pi]} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(|t|^\alpha) \text{ for } 0 < \alpha \leq 1,\ t > 0,\ r \geq 1;$$

(c) $f \in Lip(\xi(t), r)$, if

$$\|f(x+t) - f(x)\|_r = \left(\int_{[0,2\pi]} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)),\ r \geq 1,\ t > 0,$$

IJARSE
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where, $\xi(t)$ is any positive increasing function;

(d) $f \in W(L_r, \xi(t))$, if

$$\begin{aligned} \|[f(x+t)-f(x)]\sin^\beta x\|_r &= \left(\int_{[0,2\pi]} \left|[f(x+t)-f(x)]\sin^\beta x\right|^r dx\right)^{\frac{1}{r}} \\ &= O(\xi(t)), \beta \geq 0. \end{aligned}$$

Remark 2. If we take $\beta = 0$, then $W(L_r, \xi(t))$-class coincides with the class $Lip(\xi(t), r)$, if $\xi(t) = t^\alpha$, then the class $Lip(\xi(t), r)$ coincides with the class $Lip(\alpha, r)$ and if $r \to \infty$, then $Lip(\alpha, r)$ class reduces to the $Lip(\alpha)$ class.

Furthermore, as regards to the norm in L_∞ and L_r-spaces, the L_∞-norm of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}\}$$

and L_r-norm of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\|f\|_r = \left(\int_{[0,2\pi]} |f(x)|^r dx\right)^{\frac{1}{r}}, \ r \geq 1.$$

The degree of approximation of a function $f : \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial $\{t_n\}$ of order n under $\|.\|_\infty$ is defined by

$$\|t_n - f(x)\|_\infty = \sup\{|t_n(x) - f(x)| : x \in \mathbb{R}\}$$

and the degree of approximation of $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_{t_n} \|t_n - f\|_r.$$

We use the following notations throughout this paper:

$$\begin{aligned} \psi(t) &= f(x+t) + f(x-t) \ \text{ and} \\ \widetilde{K_n}(t) &= \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}. \end{aligned}$$

$$\tau = \left[\frac{1}{t}\right], \text{where } \tau \text{ the greatest integer not greater than } \frac{1}{t}.$$

III. KNOWN THEOREMS

Dealing with the product $(E,1)(C,1)$ mean, in 2011 Nigam and Sharma [7] proved the following theorem.

Theorem 1. [7] If $\bar{f}$, conjugate to a 2π-periodic function f, belongs to $W(L_r, \xi(t))$ class, then its degree of approximations by $(E,1)(C,1)$ means of conjugate Fourier series is given by

$$\|\overline{(EC)^1} - \bar{f}\|_r = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \tag{3.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left(\frac{\xi(t)}{t}\right) \text{ be a decreasing sequence,} \tag{3.2}$$

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t|\psi(t)|}{\xi(t)}\right)^r \sin^{\beta r} t dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{3.3}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{(n+1)^\delta\right\}, \tag{3.4}$$

IJARSE
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where δ is any arbitrary number such that $s(1-\delta)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$, $1\le r\le\infty$, conditions (3.3) and (3.4) hold uniformly in x, $\overline{(EC)}_n^1$ is the $(E,1)(C,1)$ means of the series (2.5) and

$$\bar{f}(x)=-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}tdt \tag{3.5}$$

Dealing with the product $(C,1)(E,q)$ mean, in 2012 Nigam and Sharma [6] proved the following theorem.

Theorem 2. [6] If a function $\bar{f}(x)$, conjugate to a 2π-periodic function $f(x)$ belonging to class $W(L_r,\xi(t))$, $r\ge 1$, then its degree of approximation by $\overline{(C,1)(E,q)}$ product means of conjugate Fourier series is given by

$$\|\overline{C_n^1E_n^q}-\bar{f}\|_r=O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \tag{3.6}$$

provided $\xi(t)$ satisfies the condition (3.7),

$$\left(\frac{\xi(t)}{t}\right) \text{ is non-increasing in t,} \tag{3.7}$$

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r\sin^{\beta r}tdt\right\}^{\frac{1}{r}}=O\left(\frac{1}{n+1}\right) \tag{3.8}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}}=O\left\{(n+1)^{\delta}\right\}, \tag{3.9}$$

where δ is any arbitrary number such that $s(1-\delta)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$, $1\le r\le\infty$, conditions (3.8) and (3.9) hold uniformly in x, $\overline{C_n^1E_n^q}$ is the $\overline{(C,1)(E,q)}$ means of the series (2.5) and

$$\bar{f}(x)=-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}tdt \tag{3.10}$$

provided

$$(1+q)^\tau\sum_{k=\tau}^{n}(1+q)^{-k}=O(n+1). \tag{3.11}$$

IV. MAIN THEOREM

The objective of this paper to prove the following theorem.

Theorem 3. If a function $\widetilde{f}(x)$, conjugate to 2π periodic function of $f(x)$ belonging to class $W(L_r,\xi(t))$, $r\ge 1$, and integrable in the Lebesgue sense in $[0,2\pi]$, then its degree of approximation by $(\overline{N},p_n,q_n)(E,s)$-summability mean of conjugate series of Fourier series is given by

$$\|\widetilde{T_n}^{\overline{N}E}-\widetilde{f}\|_r=O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}, \tag{4.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ be a decreasing sequence,} \tag{4.2}$$

$$\left\{\int_{[0,\frac{1}{n+1}]}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r\sin^{\beta r}tdt\right\}^{\frac{1}{r}}=O\left(\frac{1}{n+1}\right) \tag{4.3}$$

and

$$\left\{\int_{[\frac{1}{n+1},\pi]}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}}=O\left\{(n+1)^{\delta}\right\}, \tag{4.4}$$

where δ is any arbitrary number such that $s(1-\delta)-1>0, \frac{1}{r}+\frac{1}{s}=1$, conditions (4.3) and (4.4) hold uniformly in x, $\widetilde{T_n}^{\overline{N}E}$ is the $(\overline{N},p_n,q_n)(E,s)$ mean of conjugate series of Fourier series (2.5) and

$$\tilde{f}(x)=-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}tdt. \tag{4.5}$$

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To prove the theorem we need the following lemmas.

Lemma 1. $|\widetilde{K_n}(t)| = O(n)$, for $0 \le t \le \frac{1}{n+1}$.

Proof. For $0 \le t \le \frac{1}{n+1}$, we have $\sin\ nt \le n \sin\ t$.

$$\begin{aligned}
|\widetilde{K_n}(t)| &= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}\right| \\
&= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(\frac{\cos\frac{t}{2} - \cos vt.\cos\frac{t}{2} + \sin vt.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right)\right\}\right| \\
&= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(\frac{\cos\frac{t}{2}(2\sin^2 v\frac{t}{2})}{\sin\frac{t}{2}} + \sin vt\right)\right\}\right| \\
&\le \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(O(2\sin v\frac{t}{2}\sin v\frac{t}{2})\right) + v\sin t\right\}\right| \\
&\le \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(O(v) + O(v)\right)\right\}\right| \\
&= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}O(k)\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\right\}\right| \\
&= \frac{O(n)}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}(1+s)^k\right\}\right| \\
&= O(n).
\end{aligned}$$

Lemma 2. $|\widetilde{K_n}(t)| = O(\frac{1}{t})$, for $\frac{1}{n+1} < t \le \pi$.

Proof. For $\frac{1}{n+1} < t \le \pi$ and by using Jordans lemma, $\sin\frac{t}{2} \ge \frac{t}{\pi}$ and $\sin nt \le 1$.

$$\begin{aligned}
|\widetilde{K_n}(t)| &= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}\right| \\
&= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(\frac{\cos\frac{t}{2} - \cos vt.\cos\frac{t}{2} + \sin vt.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right)\right\}\right| \\
&\le \frac{1}{2\pi R_n}\left(\frac{\pi}{t}\right)\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(\cos\frac{t}{2}(2\sin^2 v\frac{t}{2}) + \sin vt\right)\right\}\right| \\
&\le \frac{1}{2tR_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\right\}\right| \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

V. PROOF OF MAIN THEOREM

Following Lal [2], the n^{th} partial sum $\tilde{s_n}(f)$ of the series (2.5) is given by

$$\tilde{s_n}(x) - \tilde{f}(x) = \frac{1}{2\pi}\int_{[0,\pi]}\psi(t)\frac{\cos\frac{t}{2} - \cos(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt.$$

IJARSE
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Further, using (2.5) the (E,s) transform of $\tilde{s_n}(f)$ is given by

$$\tilde{E_n^s} - \tilde{f}(x) = \frac{1}{2\pi(1+s)^n}\int_{[0,\pi]}\frac{\psi(t)}{\sin\frac{t}{2}}\left\{\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\left(\cos\frac{t}{2}-\cos\left(k+\frac{1}{2}\right)t\right)\right\}dt.$$

Now, denoting the $(\overline{N},p_n,q_n)(E,s)$ transform of $\tilde{s_n}(f)$ by $\widetilde{T_n}^{\overline{N}E}$, we have

$$\begin{aligned}\widetilde{T_n}^{\overline{N}E} - \tilde{f}(x) &= \frac{1}{2\pi R_n}\sum_{k=0}^{n}p_kq_k\int_{[0,\pi]}\frac{\psi(t)}{\sin\frac{t}{2}}\left\{\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\left(\cos\frac{t}{2}-\cos\left(v+\frac{1}{2}\right)t\right)\right\}dt\\ &= \left[\int_{[0,\frac{1}{n+1}]} + \int_{[\frac{1}{n+1},\pi]}\right]\psi(t)\widetilde{K_n}(t)dt\\ &= I_1 + I_2(\text{say}).\end{aligned}$$

Now,

$$|I_1| \le \int_{[0,\frac{1}{n+1}]}|\psi(t)||\widetilde{K_n}(t)|dt.$$

As,

$$|\psi(x,t)-\psi(x)| \le |f(u+x+t)-f(u+x)| + |f(u-x-t)-f(u-x)|,$$

so, by using Minkowski's inequality,

$$\begin{aligned}\left[\int_{[0,2\pi]}\left|\{\psi(x+t)-\psi(x)\}\sin^\beta x\right|^r dx\right]^{\frac{1}{r}} &\le \left[\int_{[0,2\pi]}\left|\{f(u+x+t)-f(u+x)\}\sin^\beta x\right|^r dx\right]^{\frac{1}{r}}\\ &+ \left[\int_{[0,2\pi]}\left|\{f(u-x-t)-f(u-x)\}\sin^\beta x\right|^r dx\right]^{\frac{1}{r}}\\ &= O(\xi(t)).\end{aligned}$$

Furthermore,

$$f \in W(L_r,\xi(t)) \Rightarrow \psi \in W(L_r,\xi(t)),$$

thus,

$$|I_1| \le \int_{[0,\frac{1}{n+1}]}\left|\frac{t\psi(t)\sin^\beta t}{\xi(t)}.\frac{\xi(t)\widetilde{K_n}(t)}{t\sin^\beta t}\right|dt.$$

Now, using Hölder's inequality and Lemma-1, we have

$$\begin{aligned}|I_1| &\le \left(\int_{[0,\frac{1}{n+1}]}\left|\frac{t\psi(t)\sin^\beta t}{\xi(t)}\right|^r dt\right)^{\frac{1}{r}}\\ &\times \left(\lim_{\varepsilon\to 0}\int_{[\varepsilon,\frac{1}{n+1}]}\left|\frac{\xi(t)\widetilde{K_n}(t)}{t\sin^\beta t}\right|^s dt\right)^{\frac{1}{s}} \text{ for some } \left(0<\varepsilon<\frac{1}{n+1}\right)\\ &= O\left(\frac{1}{n+1}\right)\left(\lim_{\varepsilon\to 0}\int_{[\varepsilon,\frac{1}{n+1}]}\left|\frac{\xi(t)O(n)}{t\sin^\beta t}\right|^s dt\right)^{\frac{1}{s}} \text{ by (4.3)}\\ &= O\left(\frac{n}{n+1}\right)\left(\lim_{\varepsilon\to 0}\int_{[\varepsilon,\frac{1}{n+1}]}\left|\frac{\xi(t)}{t\sin^\beta t}\right|^s dt\right)^{\frac{1}{s}}.\end{aligned}$$

Also, by 2nd mean value theorem, we have

$$\begin{aligned}|I_1| &= O(1)\xi\left(\frac{1}{n+1}\right)\left[\int_{[\varepsilon,\frac{1}{n+1}]}\left(\frac{1}{t^{1+\beta}}\right)^s dt\right]^{\frac{1}{s}}\\ &= O\left\{\xi\left(\frac{1}{n+1}\right)\left[\left(\frac{t^{-s(1+\beta)+1}}{-s(1+\beta)+1}\right)^{\frac{1}{s}}\right]_0^{\frac{1}{n+1}}\right\}\end{aligned}$$

IJARSE
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$$
\begin{aligned}
&= O\left\{\xi\left(\frac{1}{n+1}\right)\left[t^{-(\beta+1-\frac{1}{s})}\right]_0^{\frac{1}{n+1}}\right\}\\
&= O\left\{\xi\left(\frac{1}{n+1}\right)\left(\frac{1}{n+1}\right)^{-(\beta+\frac{1}{r})}\right\} \text{ (since } r^{-1}+s^{-1}=1)\\
&= O\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+\frac{1}{r}}\right\}. \qquad (5.1)
\end{aligned}
$$

Next,

$$|I_2| \le \int_{\frac{1}{n+1}}^{\pi}\left|\frac{t^{-\delta}\psi(t)\sin^\beta t}{\xi(t)}.\frac{\xi(t)\widetilde{K_n}(t)}{t^{-\delta}\sin^\beta t}\right|dt.$$

By using Hölder's inequality and lemma-2,

$$
\begin{aligned}
|I_2| &\le \left(\int_{[\frac{1}{n+1},\pi]}\left|\frac{t^{-\delta}|\psi(t)|\sin^\beta t}{\xi(t)}\right|^r dt\right)^{\frac{1}{r}}\left(\int_{[\frac{1}{n+1},\pi]}\left|\frac{\xi(t)\widetilde{K_n}(t)}{t^{-\delta}\sin^\beta t}\right|^s dt\right)^{\frac{1}{s}}\\
&= O\{(n+1)^\delta\}\left(\int_{[\frac{1}{n+1},\pi]}\left(\frac{\xi(t)}{t^{1-\delta+\beta}}\right)^s dt\right)^{\frac{1}{s}} \text{ by (4.4)}\\
&= \{(n+1)^\delta\}\left(\int_{[\frac{1}{\pi},n+1]}\left(\frac{\xi(\frac{1}{y})}{y^{\delta-1-\beta}}\right)^s .\frac{dy}{y^2}\right)^{\frac{1}{s}} \text{ by (4.2).}
\end{aligned}
$$

Again, by using 2nd mean value theorem,

$$
\begin{aligned}
|I_2| &= O\left\{(n+1)^\delta\xi\left(\frac{1}{n+1}\right)\right\}\left(\int_{[\frac{1}{\pi},n+1]}\frac{dy}{y^{s(\delta-1-\beta)+2}}\right)^{\frac{1}{s}}\\
&= O\left\{(n+1)^\delta\xi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{(n+1)^{s(1+\beta-\delta)-1}-\pi^{-s(1+\beta-\delta)+1}}{s(1+\beta-\delta)-1}\right\}^{\frac{1}{s}}\\
&= O\left\{(n+1)^\delta\xi\left(\frac{1}{n+1}\right)\left((n+1)^{\beta-\delta+1-\frac{1}{s}}\right)\right\}\\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}, \text{ since } (r^{-1}+s^{-1}=1). \qquad (5.2)
\end{aligned}
$$

Now, combining (5.1) and (5.2), we have

$$|\widetilde{T_n}^{NE}-\tilde{f}| = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}. \qquad (5.3)$$

Now, using L_r-norm, we have

$$
\begin{aligned}
\|\widetilde{T_n}^{NE}-\tilde{f}\|_r &= \left\{\int_0^{2\pi} O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}\\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left(\int_0^{2\pi}dx\right)^{\frac{1}{r}}\\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}.
\end{aligned}
$$

Which completes the proof of the theorem.

Corollary 1. If we put $\beta=0$ in Theorem 2, then the generalized Lipschitz $W(L^r,\xi(t))$-class reduces to $Lip(\xi(t),r)$, where, $\xi(t)$ is any positive increasing function. The degree of approximation of $\tilde{f}$, conjugate to 2π-periodic function of f, belonging to $Lip(\xi(t),r)$ class by $(\overline{N},p_n,q_n)(E,s)$-summability mean of conjugate series of Fourier series is given by

$$\|\widetilde{T_n}^{NE}-\tilde{f}\|_r = O\left\{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\ r\ge 1. \qquad (5.4)$$

Corollary 2. If we put $\beta=0$ and $\xi(t)=t^\alpha$, $0<t\le 1$ in Theorem 2, then the generalized Lipschitz $W(L^r,\xi(t))$-class reduces to $Lip(\alpha,r)$. The degree of approximation of $\tilde{f}$, conjugate 2π periodic function of

f, belonging to $Lip(\alpha, r)$ class by $(\overline{N}, p_n, q_n)(E, s)$-summability mean of conjugate series of Fourier series is given by

$$\|\widetilde{T_n}^{\overline{N}E} - \tilde{f}\|_r = O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\},\ 0<\alpha<1,\ r\geq 1. \tag{5.5}$$

Corollary 3. If we put $\beta = 0$, $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$ and $r \to \infty$ in Theorem 2, then the generalized Lipschitz $W(L^r, \xi(t))$-class reduces to $Lip(\alpha)$. The degree of approximation of $\tilde{f}$, conjugate 2π periodic functions of f, belonging to $Lip(\alpha)$ class by $(\overline{N}, p_n, q_n)(E, s)$-summability mean of conjugate series of Fourier series is given by

$$\|\widetilde{T_n}^{\overline{N}E} - \tilde{f}\|_\infty = O\left\{(n+1)^{-\alpha}\right\},\ \text{where } 0<\alpha<1. \tag{5.6}$$

VI. CONCLUSION

There are various types of results concerning the degree of approximations of periodic signals (functions) belonging to the different Lipschitz classes are reviewed. The theorem in this paper is an attempt to establish the approximation of the signal (function) belonging to the $W(L_r, \xi(t))$-class by $(\overline{N}, p_n, q_n)(E, s)$ summability mean of conjugate series of Fourier series, which is the generalization of several known theorems.

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