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# NUMBER OF ZEROS OF AN ANALYTIC FUNCTION IN A DISK

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**ABSTRACT**

*Consider an analytic function* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *. In this paper we find the number of zeros of such type of analytic functions with restricted coefficients, in a disk centered at origin*.

***Keywords: Polynomial, Analytic Function, Zeros.***

**2010 Mathematics subject classification:** 30C10, 30C15

## I. INTRODUCTION AND STATEMENT OF RESULTS

One of the basic theorem of mathematics is the Fundamental Theorem of Algebra, according to which, *"every polynomial of degree* $n \geq 1$ *has exactly* $n$ *zeros in the complex plane"*. This theorem does not however say anything regarding the location of zeros of a polynomial. The problem of locating some or all the zeros of a given polynomial as a function of its coefficients is of long standing interest in mathematics. This fact can be deduced by glancing at the references in the comprehensive books of Marden [13] and Milovanovic, Mitrinovic and Rassias [14] and by noting the abundance of recent publications on the subject.

Historically speaking, the subject dates from about the time when the geometric representation of the complex numbers was introduced into mathematics, and the first contributors to the subject were Gauss and Cauchy. Cauchy [4] improved the result of Gauss and proved:

**Theorem A:** *If* $p(z) = \sum_{j=0}^{n} a_j z^j, a_n \neq 0,$ *is a polynomial of degree* $n$ *with complex coefficients, then all the zeros of* $p(z)$ *lie in the circle,*

$$|z| \leq 1 + \max_{0 \leq j \leq n-1} \left|\frac{a_j}{a_n}\right|.$$

Other results of similar type were obtained among others by Aziz [1], [11] etc. Now we mention the following elegant result which is commonly known as Eneström-Kakeya Theorem in the theory of distribution of zeros of polynomials.

**Theorem B:** *If* $p(z) = \sum_{j=0}^{n} a_j z^j$, $a_n \neq 0$, *is a polynomial of degree* $n$ *with real coefficients satisfying* $a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_0 > 0$, *then all the zeros of* $p(z)$ *lie in the circle* $|z| \leq 1$.

Theorem B was proved by Eneström [6], independently by Kakeya [12] and Hurwitz [8].

Consider the polynomials related analytic function $f(z) = \sum_{j=0}^{\infty} a_j z^j$ not identically zero. Finding approximate zeros of polynomial related analytic functions is an important and well-studied problem. Methods to find the number of zeros of polynomial related analytic functions have already been addressed by Aziz and Mohammad [2], Dewan and Govil [5], Liman [10], Shah [16], etc.

Aziz and Mohammad [2] extended Eneström-Kakeya Theorem to the class of analytic functions $f(z) = \sum_{j=0}^{\infty} a_j z^j$ with its coefficients $a_j$ satisfying a relation analogous to the condition of Eneström-kakeya theorem.

**Theorem C:** *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *(not identically equal to zero) be analytic in* $|z| \leq t$. *If*

$a_j > 0$ *and* $a_{j-1} - t a_j \geq 0, \quad j = 0,1,2,\ldots$

*Then* $f(z)$ *does not vanish in* $|z| \leq t$.

Shah and Liman [17] considered more general class of analytic functions by assuming the coefficients to be complex numbers and generalized Theorem C in the following way.

**Theorem D:** *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *not identically equal to zero be analytic in* $|z| \leq t$ *such that for* $k \geq 1$,

$k|a_0| \geq t|a_1| \geq t^2|a_2| \geq \cdots,$

*And for some* $\beta$,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0,1,2,\ldots$$

*Then* $f(z)$ *does not vanish in*

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$$\left|z - \frac{(k-1)t}{M^2-(k-1)^2}\right| \le \frac{Mt}{M^2-(k-1)^2},$$

*Where*

$$M = k(Cos\alpha + Sin\alpha) + 2\frac{Sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j.$$

Next we consider a result in Titchmarsh's classic "The Theory of Functions", in which he states and proves the following (see page 171 of second edition) [18].

**Theorem E:** *Let $F(z)$ be analytic in $|z| \le R$ and $|F(z)| \le M$ in $|z| \le R$. If $F(0) \ne 0$, then for $0 < \delta < 1$, the number of zeros of $F(z)$ in the disc $|z| \le \delta R$ is less than*

$$\frac{1}{log\frac{1}{\delta}} log\frac{M}{|F(0)|}.$$

By putting a restriction on the coefficients of a polynomial similar to that of the Eneström-Kakeya Theorem, Mohammad [15] used a special case of Theorem D to prove the following:

**Theorem F:** *Let $p(z) = \sum_{j=0}^{\infty} a_j z^j$ be such that $0 < a_0 \le a_1 \le a_2 \le \cdots \le a_n$. Then the number of zeros in $|z| \le \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{log\ 2} log\left(\frac{a_n}{a_0}\right)$$

Concerning the number of zeros of an analytic function in a disk, recently Irshad et al [9] proved the following:

**Theorem G:** *Let $p(z) = \sum_{j=0}^{\infty} a_j z^j$ be analytic in $|z| \le 1$, if $Re\ a_j = \alpha_j$ and $Im\ a_j = \beta_j, j = 0,1,2, \ldots$ and for some finite $k$,*

$$0 < \alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \lambda a_k \ge a_{k+1} \ge \cdots$$

*With $\lambda \ge 1$, then the number of zeros of $f(z)$ in $|z| \le \frac{1}{2}$ does not exceed*

$$1+\frac{1}{\log 2}\left\{\frac{\lambda a_k+(\lambda-1)|a_k|+\sum_{j=0}^{\infty}|\beta_j|}{|a_0|}\right\}.$$

In this paper we first present a generalizations of Theorem G and prove some more results concerning the number of zeros of polynomial related analytic function in a disk which in turn generalizes Theorem F.

**Theorem 1:** *Let* $f(z)=\sum_{j=0}^{\infty}a_j z^j$ *not identically zero, be analytic in* $|z|\leq t$*, where* $a_0\neq 0$. *If for some* $t>0, 0\leq k\leq n$*,*

$|a_0|\leq t|a_1|\leq\cdots\leq t^{k-1}|a_{k-1}|\leq t^k|a_k|\geq t^{k+1}|a_{k+1}|\geq\cdots\geq t^n|a_n|\geq\cdots$

*and for some real* $\alpha$ *and* $\beta$*,*

$$\left|\arg a_j-\beta\right|\leq\alpha\leq\frac{\pi}{2}, j=0,1,2,\ldots$$

*Then for* $0<\delta<1$*, the number of zeros of* $f(z)$ *in* $|z|\leq\delta t$ *does not exceed*

$$\frac{1}{\log {}^{1}/_{\delta}}\log\frac{M_1}{|a_0|},$$

*Where*

$$M_1=2t^{k+1}|a_k|\cos\alpha+2\sin\alpha\sum_{i=0}^{\infty}|a_i|t^{i+1}-t|a_0|(\sin\alpha+\cos\alpha-1).$$

Now we put restriction on the real part of the coefficients of a complex polynomial and prove the following result.

**Theorem 2:** *Let* $f(z)=\sum_{j=0}^{\infty}a_j z^j$ *(not identically zero), be analytic in* $|z|\leq t$*, with* $Re(a_j)=\alpha_j$ *and* $Im(a_j)=\beta_j$*. If for some* $k$ *with* $0\leq k\leq n$*,*

$$\alpha_0\leq t\alpha_1\leq\cdots\leq t^{k-1}\alpha_{k-1}\leq t^k\alpha_k\geq t^{k+1}\alpha_{k+1}\geq\cdots\geq t^n\alpha_n\geq\cdots.$$

*Then for* $0<\delta<1$*, the number of zeros of* $f(z)$ *in* $|z|\leq\delta t$ *does not exceed*

$$\frac{1}{\log {}^{1}/_{\delta}} \log \frac{M_2}{|a_0|},$$

*Where*

$$M_2 = t(|\alpha_0| - \alpha_0) + 2t^{k+1}\alpha_k + 2\sum_{j=0}^{\infty}|\beta_j|t^{j+1}.$$

**Remarks:** If we take $\alpha_j$ and $\beta_j$ to be positive for all j and $t = 1$, then Theorem 2 reduces to a special case of Theorem G.

Finally, we put restriction on real as well as on imaginary parts of the coefficients of a complex polynomial and prove the following results.

**Theorem 3:** *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *(not identically zero), be analytic in* $|z| \leq t$, *with* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$. *If for some* $k$ *with* $0 \leq k \leq n$,

$$\alpha_0 \leq t\alpha_1 \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^n\alpha_n \geq \cdots$$

*And for some* $l$ *with* $0 \leq l \leq n$,

$$\beta_0 \leq t\beta_1 \leq \cdots \leq t^{l-1}\beta_{k-1} \leq t^l\beta_k \geq t^{l+1}\beta_{k+1} \geq \cdots \geq t^n\beta_n \geq \cdots.$$

*Then the number of zeros of* $f(z)$ *in* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log {}^{1}/_{\delta}} \log \frac{M_3}{|a_0|},$$

*Where*

$$M_3 = t(|\alpha_0| - \alpha_0) + t(|\beta_0| - \beta_0) + 2t^{k+1}\alpha_k + 2t^{l+1}\beta_l.$$

**Remark 2:** If we take $\alpha_j$ *and* $\beta_j$ to be positive for all $j$ and $t = 1$, then Theorem 3 reduces to a result earlier proved by Irshad et al [9].

**Lemma**

For the proof of some of these results we need the following lemma which is due to Govil and Rahman [7].

**Lemma 1:** For any two complex numbers $b_0$ *and* $b_1$ such that $|b_0| \geq |b_1|$ and

$$|arg\ b_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \qquad j = 0,1,2,\ldots$$

for some real $\beta$, then

$$|b_0 - b_1| \leq (|b_0| - |b_1|)\cos\alpha + (|b_0| + |b_1|)\sin\alpha\,.$$

**Proof of the Theorems:**

**Proof of Theorem 1.** Consider the polynomial

$$g(z) = (t - z)f(z)$$

$$= (t - z)(a_0 + a_1 z + a_2 z^2 + \cdots)$$

$$= ta_0 + \sum_{i=1}^{\infty}(ta_i - a_{i-1})z^i.$$

Therefore for $|z| \leq t$, we have

$$|g(z)| \leq t|a_0| + \sum_{i=1}^{\infty}|ta_i - a_{i-1}|t^i$$

$$= t|a_0| + \sum_{i=1}^{k}|ta_i - a_{i-1}|t^i + \sum_{i=k+1}^{\infty}|a_{i-1} - ta_i|t^i$$

Now $|a_0| \leq t|a_1| \leq \cdots \leq t^{k-1}|a_{k-1}| \leq t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^n|a_n| \geq \cdots$.

Therefore by Lemma 1, we get

$$|g(z)| \leq t|a_0| + \sum_{i=1}^{k}\{(t|a_i| - |a_{i-1}|)\cos\alpha + (t|a_i| + |a_{i-1}|)\sin\alpha\}t^i$$
$$+ \sum_{i=k+1}^{\infty}\{(|a_{i-1}| - t|a_i|)\cos\alpha + (|a_{i-1}| + t|a_i|)\sin\alpha\}t^i$$

$$= t|a_0| + \sum_{i=1}^{k}\{(t|a_i| - |a_{i-1}|)\cos\alpha\}t^i \quad + \sum_{i=k+1}^{\infty}\{(|a_{i-1}| - t|a_i|)\cos\alpha\}t^i$$

$$+ \sum_{i=1}^{\infty}\{(t|a_i| + |a_{i-1}|)\sin\alpha\}t^i$$

$$= t|a_0| + 2t^{k+1}|a_k|\cos\alpha - t|a_0|\cos\alpha + t|a_0|\sin\alpha + 2\sin\alpha\sum_{i=1}^{\infty}|a_i|t^{i+1}$$

$$= 2t^{k+1}|a_k|\cos\alpha + 2\sin\alpha\sum_{i=0}^{\infty}|a_i|t^{i+1} - t|a_0|(\sin\alpha + \cos\alpha - 1)$$

$$= M_1(say)$$

Now $g(z)$ is analytic in $|z| \leq t$ and $|g(z)| \leq M_1$ for $|z| \leq t$. Moreover $g(0) = a_0$. Therefore by Theorem E, the number of zeros of $g(z)$ (and hence of $f$) in $|z| \leq t\delta$ does not exceed

$$\frac{1}{log\, {}^{1}/_{\delta}} log\, \frac{M_1}{|a_0|},$$

Where $M_1 = 2t^{k+1}|a_k|\cos\alpha + 2\sin\alpha\sum_{i=0}^{\infty}|a_i|t^{i+1} - t|a_0|(\sin\alpha + \cos\alpha - 1)$.

This completes proof of Theorem 1.

**Proof of Theorem 2:** Consider the polynomial

$$g(z) = (t - z)f(z)$$

$$= (t - z)(a_0 + a_1 z + a_2 z^2 + \cdots)$$

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$$= t\alpha_0 + \sum_{i=1}^{\infty}(t\alpha_i - a_{i-1})z^i.$$

$$= t(\alpha_0 + i\beta_0) + \sum_{i=1}^{\infty}(t\alpha_i - \alpha_{i-1})z^i + i\sum_{i=1}^{\infty}(t\beta_i - \beta_{i-1})z^i.$$

Therefore for $|z| \leq t$, we have

$$|g(z)| \leq t(|\alpha_0| + |\beta_0|) + \sum_{i=1}^{\infty}|t\alpha_i - \alpha_{i-1}|t^i + \sum_{i=1}^{\infty}(t|\beta_i| + |\beta_{i-1}|)t^i$$

$$= t(|\alpha_0| + |\beta_0|) + \sum_{i=1}^{k}(t\alpha_i - \alpha_{i-1})t^i + \sum_{i=k+1}^{\infty}(\alpha_{i-1} - t\alpha_i)t^i + \sum_{i=1}^{\infty}(t|\beta_i| + |\beta_{i-1}|)t^i$$

$$= t(|\alpha_0| + |\beta_0|) + \sum_{i=1}^{k}(t\alpha_i - \alpha_{i-1})t^i + \sum_{i=k+1}^{\infty}(\alpha_{i-1} - t\alpha_i)t^i + 2\sum_{i=1}^{\infty}|\beta_i|t^{i+1} + t|\beta_0|$$

$$= t(|\alpha_0| + |\beta_0|) + 2t^{k+1}\alpha_k - t\alpha_0 + 2\sum_{i=1}^{\infty}|\beta_i|t^{i+1} + t|\beta_0|$$

$$= t(|\alpha_0| - \alpha_0) + 2t^{k+1}\alpha_k + 2\sum_{i=0}^{\infty}|\beta_i|t^{i+1}.$$

$$= M_2(say)$$

Proceeding in the same lines as in the proof of Theorem 1, the proof of this result follows.

**Proof of Theorem 3:** As in the proof of Theorem 2, we have

$$g(z) = t(\alpha_0 + i\beta_0) + \sum_{i=1}^{\infty}(t\alpha_i - \alpha_{i-1})z^i + i\sum_{i=1}^{\infty}(t\beta_i - \beta_{i-1})z^i.$$

Therefore for $|z| \leq t$, we get

$$|g(z)| \leq t(|\alpha_0| + |\beta_0|) + \sum_{i=1}^{\infty} |t\alpha_i - \alpha_{i-1}| t^i + \sum_{i=1}^{\infty} |t\beta_i - \beta_{i-1}| t^i$$

$$= t(|\alpha_0| + |\beta_0|) + \sum_{i=1}^{k} (t\,\alpha_i - \alpha_{i-1}) t^i$$

$$+ \sum_{i=k+1}^{\infty} (\alpha_{i-1} - t\alpha_i) t^i + \sum_{i=1}^{l} (t\beta_i - \beta_{i-1}) t^i + \sum_{i=l+1}^{\infty} (\beta_{i-1} - t\beta_i) t^i$$

$$= t(|\alpha_0| + |\beta_0|) + 2t^{k+1}\alpha_k - t\alpha_0 + 2t^{l+1}\beta_l - t\beta_0$$

$$= t(|\alpha_0| - \alpha_0) + t(|\beta_0| - \beta_0) + 2t^{k+1}\alpha_k + 2t^{l+1}\beta_l$$

$$= M_3(say)$$

The result now follows as in the proof of Theorem 1.

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