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Captivating Cattle problem

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ABSTRACT

One of the most profound problem in mathematics of antiquity was posed by Archimedes, arguably the greatest scientist of ancient times. The problem is so posed in a way that its solution is far away to guess a solution. This problem is called "Cattle Problem" in mathematics literature. Both from a computational and from a historical perspective, the cattle problem of Archimedes is considered to be one of the best problems posed in mathematics. In this paper, we present the history of the problem, state the problem from the original source and discuss its solution in a detailed fashion thereby proving that this interesting problem has infinitely many solutions.

Keywords: Triangular Numbers, Square Numbers, Pell's Equation.

I. HISTORY OF THE CATTLE PROBLEM

Archimedes' cattle problem (or the problema bovinum or problema Archimedis) is a problem arising in Diophantine Analysis, which is a study of solving polynomial equations with integer solutions. Attributed to Archimedes of Syracuse, the problem involves computing the number of cattle in a herd of the sun god from a given set of constraints. The problem was discovered by Gotthold Ephraim Lessing in a Greek manuscript containing a poem of forty-four lines, in the Herzog August Libraryin Wolfenbüttel, Germanyin 1773.

The problem asks for the number of white, black, dappled and brown bulls and cows belonging to the sun god, subject to several arithmetical conditions. A version in English heroic couplets, published in **[1]**, is provided below.

II.POSING THE PROBLEM

That Archimedes conceived in verse and posed to the specialists at Alexandria in letter to Eratosthenes of Cyrene.

"*The Sun god's cattle, friend, apply thy care to count their number, hast thou wisdom's share. They grazed of old on the Thrinacian floor of sic'ly's island , herded into four, colour by colour: one herd white as cream, the next in coats glowing with ebon gleam, brown- skinned the third , and stained with spots the last Each herd saw bulls in power unsurpassed, in ratios these: count half the ebon- hued and one third more, then all the brown include ; thus, friend, canst thouthe white bulls number tell. The ebon did the brown exceed as well, now by a fourth and fifth part of the stained. To know the spotted - all bulls that remained - reckon again the brown bulls and unite these with a sixth and seventh of the white Among the cows, the tale of silver - haired was, when with bulls and cows of black compared exactly one in three plus one in four. The black cows counted one in four once more, plus now a fifth, of the bespeckled breed when, bulls withal, they wandered out to feed. The speckled cows tallied a fifth and sixth of all the brown-haired, males and females mixed. Lastly, the brown cows numbered half a third and one in seven of the silver herd. Tell'st thou unfailingly how many head the sun possessed, o friend, both bulls well- fed and cows of every colour - no-one will deny that thou hast number art and skill, tough not yet dost thou rank among the wise. But come! also the following recognise. whene'er the sun god's white bulls joined the black, their multitude would gather in a pack of equal length and breadth, and squarely throng Thrinacia's territory broad and long. But when the brown bulls mingled with the flecked, in rows growing from one would they collect, forming a perfect triangle, with ne'er a different - colored bull and none to spare. Friend, canst thou analyse this in thy mind, and of these masses all the measures find, go forth in glory! be assured all deem thy wisdom in this discipline supreme!*"

III.SOLVING THE PROBLEM

The first part of the problem describes the proportions of bulls of the four different colours.

Let the four variables*x, y, z, t*represents the number of white, black, dappled and brown bulls respectively. Then we get the three following equations:

$$x=\left(\frac{1}{2}+\frac{1}{3}\right)y+t=\frac{5}{6}y+t \rightarrow (1)$$

$$y=\left(\frac{1}{4}+\frac{1}{5}\right)z+t=\frac{9}{20}z+t \rightarrow (2)$$

$$z=\left(\frac{1}{6}+\frac{1}{7}\right)x+t=\frac{13}{42}x+t \rightarrow (3)$$

The second part of the problem introduces four more variables x^1, y^1, z^1, t^1 and four more linear equations.

$$x^1 = \left(\frac{1}{3}+\frac{1}{4}\right)\left(y+y^1\right) = \frac{7}{12}\left(y+y^1\right) \rightarrow (4)$$

$$y^1 = \left(\frac{1}{4}+\frac{1}{5}\right)\left(z+z^1\right) = \frac{9}{20}\left(z+z^1\right) \rightarrow (5)$$

$$z^1 = \left(\frac{1}{5}+\frac{1}{6}\right)\left(t+t^1\right) = \frac{11}{30}\left(t+t^1\right) \rightarrow (6)$$

$$t^1 = \left(\frac{1}{6}+\frac{1}{7}\right)\left(x+x^1\right) = \frac{13}{42}\left(x+x^1\right) \rightarrow (7)$$

The third part of the problem require that $x+y$ be a square number and $z+t$ be a triangular number.

$$(1) \Rightarrow x = \frac{5}{6}y + t$$

$$= \frac{5}{6}\left(\frac{9}{20}z + t\right) + t$$

$$= \frac{5}{6}\left(\frac{9}{20}\left(\left(\frac{1}{6}+\frac{1}{7}\right)x + t\right) + t\right) + t$$

$$= \frac{5}{6}\left(\frac{9}{20} \times \frac{13}{42}x + \frac{9}{20}t + t\right) + t$$

$$= \frac{5}{6} \times \frac{9}{20} \times \frac{13}{42}x + \frac{5}{6} \times \frac{9}{20}t + \frac{5}{6}t + t$$

$$x - \frac{5}{6} \times \frac{9}{20} \times \frac{13}{42}x = \frac{5}{6} \times \frac{9}{20}t + \frac{5}{6}t + t$$

$$x\left(1 - \frac{5}{6} \times \frac{9}{20} \times \frac{13}{42}\right) = \left(\frac{45+100+120}{6 \times 20}\right)t$$

$$x\left(1 - \frac{585}{6 \times 20 \times 42}\right) = \left(\frac{45+100+120}{6 \times 20}\right)t$$

$$x\left(\frac{5040-585}{6 \times 20 \times 42}\right) = \frac{265t}{6 \times 20}$$

$$x\left(\frac{4455}{42}\right) = 265t$$

$$4455x = 11130t$$

IJARSE
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$$891x = 2226t$$

$$\frac{x}{2226} = \frac{t}{891} = m \ \ (say)$$

$$\Rightarrow x = 2226m$$

$$\Rightarrow t = 891m$$

$$(3) \Rightarrow z = \frac{13}{42}x + t$$

$$= \frac{13}{42}(2226m) + \ 891m$$

$$= (13 \times 53)m + 891m$$

$$z = 1580m$$

$$(2) \Rightarrow y = \frac{9}{20}z + t$$

$$= \frac{9}{20}(1580)m + 891m$$

$$y = 1602m$$

Therefore, the general solution to the first three equations is given by

$$(x,y,z,t) = m \times (2226,1602,1580,891)$$

$$(4) \Rightarrow x' = \frac{7}{12}(y + y')$$

$$= \frac{7}{12}\left(1602m + \left(\frac{9}{20}(z + z')\right)\right)$$

$$= \frac{7}{12}\left(1602m + \left(\frac{9}{20}\left(1580m + \frac{11}{30}\left(891m + \frac{13}{42}(2226m + x')\right)\right)\right)\right)$$

IJARSE
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$$=\frac{7}{12}\left(1602m+\frac{9}{20}(1580m)+\frac{9}{20}\times\frac{11}{30}(891m)+\frac{9}{20}\times\frac{11}{30}\times\frac{13}{42}(2226m)+\frac{9}{20}\times\frac{11}{30}\times\frac{13}{42}(x')\right)$$

$$=\frac{7}{12}(1602m)+\frac{7}{12}\times\frac{9}{20}(1580n)+\frac{7}{12}\times\frac{9}{20}\times\frac{11}{30}(891m)+\frac{7}{12}\times\frac{9}{20}\times\frac{11}{30}\times\frac{13}{42}(2226m)+\frac{7}{12}\times\frac{9}{20}\times\frac{11}{30}\times\frac{13}{42}x'$$

$$x'\left(1-\frac{7}{12}\times\frac{9}{20}\times\frac{11}{30}\times\frac{13}{42}\right)=\frac{282592800m+125420400m+25933446m+20054034m}{12\times20\times30\times42}$$

$$x'\left(\frac{302400-9009}{12\times20\times30\times42}\right)=\frac{454000680m}{12\times20\times30\times42}$$

$$x'\left(\frac{293391}{12\times20\times30\times42}\right)=\frac{454000680m}{12\times20\times30\times42}$$

$$x'(293391)=454000680m$$

$$\div 63$$

$$4657x'=7206360m$$

$$\frac{x'}{7206360}=\frac{m}{4657}=k\ (say) \qquad (8)$$

$$(8)\Longrightarrow\frac{x'}{7206360}=k$$

$$x'=7206360k$$

$$(8)\Longrightarrow\frac{m}{4657}=k$$

$$m = 4657k$$

$$(7) \Rightarrow t' = \frac{13}{42}(x + x')$$

$$= \frac{13}{42}(2226m + 7206360k)$$

$$= \frac{13}{42}(2226(4657k) + 7206360k)$$

$$= \frac{13}{42}(17572842k)$$

$$= 5439213k$$

$$(6) \Rightarrow z' = \frac{11}{30}(t + t^1)$$

$$= \frac{11}{30}(891m + 5439213k)$$

$$= \frac{11}{30}(891(4657k) + 5439213k)$$

$$= \frac{11}{30}(9588600)k$$

$$= 3515820k$$

$$(5) \Rightarrow y' = \frac{9}{20}(Z + Z^1)$$

$$= \frac{9}{20}(1580m + 3515820k)$$

$$= \frac{9}{20}(1580(4657k) + 3515820k)$$

$$= \frac{9}{20}(10873880)k$$

$$= 4893246k$$

The general solution to the equation (4),(5), (6) &(7) is given by

$$(x',y',z',t') = k(7206360, 4893246, 3515820, 5439213)$$

Thus, the least possible solution satisfying the first seven equations is obtained when $k = 1$ providing $m = 4657$. They are given by

$$x = 10366482, y = 7460514, z = 7358060, t = 4149387$$
$$x^1 = 7206360, y^1 = 4893246, z^1 = 3515820, t^1 = 5439213$$

Now to satisfy the third part of the problem, we choose k such that

$$x + y = (2226m + 1602m)$$
$$= 3828m$$
$$= 3828(4657k) \quad is \ \ a\ square\ and$$
$$z + t \ = 1580m + 891m$$
$$= 2471m$$

$= 2471\ (4657)k$ is a triangle number.

From the factorization,

3828×4657=2^2×3×11×29×4657 one sees that the first condition is equivalent to $k = al^2$.

where a =3×11×29×4657 and l is an integer.

Now, $z + t$ is a triangular number if and only if $8(z + t) + 1$ is a square.

We are led to the equation

$$h^2 = 8(z + t) + 1$$
$$= 8 \times 4657 \times 2471 \times al^2 + 1$$

which is the Pell equation

$$h^2 = dl^2 + 1\ for\ d = 8 \times 3 \times 11 \times 29 \times 2471 \times (4657)^2$$
$$= 410286423278424$$

Thus by Lagrange's theorem the cattle problem admits infinitely many solutions.

IV.OTHER SOLUTIONS

In 1867, the otherwise unknown German mathematician C.F. Meyer set out to solve the equation by the continued fraction method. After 240 steps in the continued fraction expansion for $\sqrt{d}$ he had still not detected the period, and he gave up.He may have been a little impatient. It was later discovered that the period length equals 203254.

The first to solve the cattle problem in a satisfactory way was A.Amthor in 1880. Amthor did not directly apply the continued fraction method. Nor did he spell out the decimal digits of the fundamental solution to the Pell equation or the corresponding solution of the cattle problem.

He did show that, in the smallest solution to the cattle problem, the total number of cattle is given by a number of 206545 digits. The full number occupies forty- seven pages of computer printout, reproduced in reduced size on twelve pages of the Journal of Recreational Mathematics. In abbreviated form it reads: 77602714...237983357...55081800. each of the six dots representing 34420 omitted digits.

Several nineteenth century German scholars were worried that so many bulls and cows might not fit on the island of Sicily, contradicting lines 3 and 4 of the poem. But as Lessing remarked, the sun god, to whom the cattle belonged, will have coped with it.

A wonderful problem from Antiquity has provided so much of pleasure in working out its solutions. This is an example of a classic problem in mathematics which has infinitely many solutions. But we don't know if ancients knew these solutions.

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