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Stability Analysis of an SIQS Model with Saturated Incidence Rate

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ABSTRACT

One of the effective methods to control the spread of the disease is to keep some of the highly infected people in isolation for some time. The word quarantine means isolating some infectives from the infectious class. In this paper an SIQS epidemic model is introduced with saturated incidence rate and quarantine. We have obtained threshold R_q which analyze the effect of the quarantine and outcomes of the disease. If R_q is less than 1, then the disease-free equilibrium exists and is globally stable i.e. disease dies out. If R_q is greater than 1, then there exists a unique positive equilibrium and is globally stable i.e. disease becomes endemic. The stability of the both equilibrium are proved by Routh-Hurwitz criteria, Lyapunov function and Dulac criteria.

KEYWORDS: Endemic, Reproduction number, Saturated incidence, Stability, Quarantine.

I. INTRODUCTION

The most commonly used effective methods to control the spread of disease are vaccination, treatment and quarantine. In order to reduce transmission of infection from infective class to susceptible class, one intervention procedure is to isolate some infective individuals. The word quarantine means to say about the forcefully or voluntarily isolating some infective individuals from the infective class. In the SIQS model infectious do not confer immunity, some of the susceptible becomes infected and then some infectives remains in the infectious class I for their whole infectious period before they return to the susceptible class S, while other infective individuals are transferred in to a quarantine class Q.

Feng et al [1, 2, 3, 4] formulated some quarantine models on disease transmission with different reaction incidence rate and studied the effect of quarantine. Hethcote et al [5] analyzed effect of quarantine in six SIQS and SIQR endemic models for infectious diseases with simple mass action, standard and quarantine-adjusted incidence rates. The incidence in an epidemic model is the rate at which susceptible becomes infectious. Number of incidence rates such as bilinear, nonlinear, standard, saturated, nonmonotone etc. have been defined and studied by different authors in epidemic models and presented a thorough qualitative analysis of the models [6, 7, 8, 9, 10, 11].

In this paper, we have considered an SIQS epidemic model with quarantine and saturated incidence rate $g(S)I = \frac{\lambda SI}{1+\alpha S}$ proposed by Anderson and May [7] and also used by Gao S et al [9] Wang and Jiang [12]. First we obtained the disease-free and the endemic equilibrium and then discuss the stability at these equilibrium points. Finally, in support of theoretical analysis the numerical simulations are carried out.

II. MATHEMATICAL MODEL

The model is given by the following non-linear ordinary differential equations:

$$\left.\begin{aligned} \frac{dS}{dt} &= b - \frac{\lambda SI}{1+\alpha S} - dS + \gamma I + \theta Q \\ \frac{dI}{dt} &= \frac{\lambda SI}{1+\alpha S} - (\gamma + \phi + d + d_1)I \\ \frac{dQ}{dt} &= \phi I - (\theta + d + d_1)Q \end{aligned}\right\} \quad (2.1)$$

Where parameters $b, d, \lambda, \phi, \theta, \gamma, d_1$ and α are positive constants. The constant b is the recruitment rate (including births and immigrations)of susceptible individuals, d is the per capita natural death rate, λ is the average number of adequate contacts (sufficient for transmission) of a person per unit time, ϕ is the rate constant for individuals leaving the infective compartment I for the quarantine compartment Q, d_1 is the disease related death rate constant in compartments I and Q, and γ and θ are the rates at which individuals recover and return to susceptible compartment S from compartments I and Q respectively, α is the parameter measures the proper prevention taken by susceptible population for epidemic control.

Total population size $N = S + I + Q$ i.e. $N' = S' + I' + Q'$

$\Rightarrow N' = b - dN - d_1(I + Q)$

In the absence of disease $N' = b - dN$ .This gives $N = \frac{b}{d} + N_0 e^{-dt}$. As $t \to \infty, N \to \frac{b}{d}$ i.e. population size N approaches to the carrying capacity $\frac{b}{d}$. It follows that the solution of (2.1) exists in the region defined by $\Omega = \left\{(S, I, Q) \in R_3^+ : S, I, Q \geq 0, S + I + Q \leq \frac{b}{d}\right\}$

III. EQILIBRIUM POINTS

For equilibrium points of (2.1), we have

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$$\left.\begin{aligned} b-\frac{\lambda SI}{1+\alpha S}-dS+\gamma I+\theta Q=0 \\ \frac{\lambda SI}{1+\alpha S}-(\gamma+\phi+d+d_1)I=0 \\ \phi I-(\theta+d+d_1)Q=0 \end{aligned}\right\} \qquad (3.1)$$

The system (2.1) always has the disease-free equilibrium point $E_0=\left(\frac{b}{d},0,0\right)$.

Define the threshold $R_q=\frac{\lambda b}{(b\alpha+d)(\gamma+\phi+d+d_1)}$ as the average number of secondary infections when one infective is entered into totally susceptible population. The number R_q is called the quarantine reproduction number and can be derived by next generation method [13]. Here we are using the term 'quarantine reproduction number' for the threshold, because in model the quarantine process is used to control the disease.

Solving (3.1) we obtain an endemic equilibrium point $E^*=(S^*,I^*,Q^*)$ of the system (2.1) in Ω when $R_q>1$, where

$$S^*=\frac{b}{\alpha b(R_q-1)+dR_q} \quad , \quad I^*=\frac{b(\theta+d+d_1)(b\alpha+d)[R_q-1]}{(d+d_1)(\theta+d+d_1+\phi)[\alpha b(R_q-1)+dR_q]} \quad , \quad Q^*=\frac{\phi I^*}{\theta+d+d_1}$$

Note that $N^*=S^*+I^*+Q^*=\frac{bd_1}{(d+d_1)[\alpha b(R_q-1)+dR_q]}+\frac{b}{d+d_1}$

It is clear that when the disease-related death rate constant $d_1=0$, the total population N^* at the endemic equilibrium E^* approaches to the disease-free carrying capacity $\frac{b}{d}$.

IV. LOCAL STABILITY ANALYSIS

Theorem (4.1): If $R_q<1$, then the disease free equilibrium E_0 of the system (2.1) is locally asymptotically stable. If $R_q>1$, then the equilibrium E_0 is unstable, the disease persists and the positive equilibrium E^* is locally asymptotically stable in the region Ω.

Proof: The variation matrix of the system (2.1) is

$$V=\begin{pmatrix} -\frac{\lambda I}{(1+\alpha S)^2}-d & -\frac{\lambda S}{(1+\alpha S)}+\gamma & \theta \\ \frac{\lambda I}{(1+\alpha S)^2} & \frac{\lambda S}{(1+\alpha S)}-(\gamma+\phi+d+d_1) & 0 \\ 0 & \phi & -(\theta+d+d_1) \end{pmatrix} \qquad (4.1)$$

At disease-free equilibrium $E_0=\left(\frac{b}{d},0,0\right)$, the variation matrix (4.1) becomes

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$$V(E_0)=\begin{pmatrix} -d & -\frac{\lambda b}{(b\alpha+d)}+\gamma & \theta \\ 0 & \frac{\lambda b}{(b\alpha+d)}-(\gamma+\phi+d+d_1) & 0 \\ 0 & \phi & -(\theta+d+d_1) \end{pmatrix}$$

The characteristic equation of $V(E_0)$ is

$$\begin{vmatrix} -d-\lambda_1 & -\frac{\lambda b}{b\alpha+d}+\gamma & \theta \\ 0 & \frac{\lambda b}{b\alpha+d}-(\gamma+\phi+d+d_1)-\lambda_1 & 0 \\ 0 & \phi & -(\theta+d+d_1)-\lambda_1 \end{vmatrix}=0$$

$$\Rightarrow(\lambda_1+d)\{\lambda_1+(\theta+d+d_1)\}\left[\lambda_1-\frac{\lambda b}{(b\alpha+d)}+(\gamma+\phi+d+d_1)\right]=0$$

Clearly the two eigen values are negative and third will also be negative if $\frac{\lambda b}{(b\alpha+d)}<(\gamma+\phi+d+d_1)$ i.e. $R_q<1$. Hence the disease-free equilibrium E_0 is locally asymptotically stable if $R_q<1$ and unstable if $R_q>1$.

Now, at the endemic equilibrium $E^*=(S^*,I^*,Q^*)$ when $R_q>1$, the variation matrix (4.1) is

$$V(E^*)=\begin{pmatrix} -\frac{\lambda I^*}{(1+\alpha S^*)^2}-d & -\frac{\lambda S^*}{(1+\alpha S^*)}+\gamma & \theta \\ \frac{\lambda I^*}{(1+\alpha S^*)^2} & \frac{\lambda S^*}{(1+\alpha S^*)}-(\gamma+\phi+d+d_1) & 0 \\ 0 & \phi & -(\theta+d+d_1) \end{pmatrix}$$

$$V(E^*)=\begin{pmatrix} -V_1-d & -V_2+\gamma & \theta \\ V_1 & V_2-(\gamma+\phi+d+d_1) & 0 \\ 0 & \phi & -(\theta+d+d_1) \end{pmatrix}$$

Where, $V_1=\frac{\lambda I^*}{(1+\alpha S^*)^2}$ and $V_2=\frac{\lambda S^*}{(1+\alpha S^*)}$

The characteristic equation of $V(E^*)$ is

$$\begin{vmatrix} -V_1-d-\lambda_1 & -V_2+\gamma & \theta \\ V_1 & V_2-(\gamma+\phi+d+d_1)-\lambda_1 & 0 \\ 0 & \phi & -(\theta+d+d_1)-\lambda_1 \end{vmatrix}=0$$

i.e. $(\lambda_1+V_1+d)(\lambda_1+\theta+d+d_1)(\lambda_1+\gamma+\phi+d+d_1-V_2)+V_1[(V_2-\gamma)(\lambda_1+\theta+d+d_1)-\theta\phi]=0$

Putting $K=\theta+d+d_1$, $D=\gamma+\phi+d+d_1$, $M=V_1+d$ and solving, we get

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$\lambda_1^3 + a_1\lambda_1^2 + a_2\lambda_1 + a_3 = 0$, Where

$a_1 = M + K + D - V_2 = K + D + d + (V_1 - V_2)$

$a_2 = MK + (M + K)(D - V_2) + V_1(V_2 - \gamma) = d(K + D) + KD + V_1(d_1 + \phi) + (V_1 - V_2)(K + d)$

$a_3 = MK(D - V_2) - V_1\theta\phi + V_1K(V_2 - \gamma)] = (V_1 - V_2)dK + V_1[d_1K + \phi(d + d_1)] + dKD$

$a_1a_2 - a_3 = (K + D)[d(d + K + D) + KD] + (V_1 - V_2)[(d + K)(d + K + 2D) + V_1(d_1 + \phi)]$
$+ (V_1 - V_2)^2(d + K) + V_1[\theta\phi + (d_1 + \phi)(d + D)]$

Clearly $a_1, a_2, a_3 > 0$ and $a_1a_2 - a_3 > 0$ provided $V_1 > V_2$. Hence by Routh-Hurwitz criteria the endemic equilibrium E^* is locally asymptotically stable if $R_q > 1$.

V. GLOBAL STABILITY ANALYSIS

Theorem 5.1: If $R_q < 1$, then the disease-free equilibrium E_0 of the system (2.1) is globally asymptotically stable in the region Ω. If $R_q > 1$ and $d_1 = 0$ (i.e. there are no disease-related deaths), then the endemic equilibrium E^* is globally asymptotically stable in the region $\Omega - \{(S, I, Q) : I = 0\}$.

Proof: First we prove the global stability at the disease-free equilibrium E_0 when $R_q < 1$.Consider a Lyapunov function $L = I$. Then the Lyapunov derivative will be

$L' = I' = \left[\frac{\lambda S}{1 + \alpha S} - (\gamma + \phi + d + d_1)\right] I \le \left[\frac{\lambda b}{d + \alpha b} - (\gamma + \phi + d + d_1)\right] I \le 0$, since $S \le \frac{b}{d}$ and $R_q < 1$.Thus if $R_q < 1$, then $L' \le 0$. Note that, $L' = 0$ if and only if $S = \frac{b}{d}, I = 0$ and $Q = 0$. Therefore the largest positive invariant set in $\{(S, I, Q) \in \Omega : L' = 0\}$ is the singleton $\{E_0\}$, where E_0 is the disease-free equilibrium. Thus by Lasalle's invariant principle [14], E_0 is globally asymptotically stable in Ω.

In order to prove the global stability of E^* when $R_q > 1$ and $d_1 = 0$ (i.e. there are no disease-related deaths), first note that $N' = b - dN$. This gives $N \to \frac{b}{d}$ as $t \to \infty$. In this case the limit system of (2.1) is given by

$$\left.\begin{aligned} N' &= 0 \\ I' &= \left[\frac{\lambda\left(\frac{b}{d} - I - Q\right)}{1 + \alpha\left(\frac{b}{d} - I - Q\right)} - (\gamma + \phi + d)\right] I \\ Q' &= \phi I - (\theta + d)Q \end{aligned}\right\} \qquad (4.2)$$

Now, we discuss in the first quadrant of IQ-Plane. Using Dulac's criteria with multipliers $D_1 = \frac{1}{I}$,

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Let $F_1 = \left[\frac{\lambda\left(\frac{b}{d} - I - Q\right)}{1+\alpha\left(\frac{b}{d} - I - Q\right)} - (\gamma + \phi + d) \right] I$, $F_2 = \phi I - (\theta + d)Q$,

then $D_1F_1 = \left[\frac{\lambda\left(\frac{b}{d} - I - Q\right)}{1+\alpha\left(\frac{b}{d} - I - Q\right)} - (\gamma + \phi + d) \right]$, $D_1F_2 = \phi - (\theta + d)\frac{Q}{I}$

We have

$$\frac{\partial}{\partial I}(D_1F_1) + \frac{\partial}{\partial Q}(D_1F_2) = -\frac{\lambda}{\left[1+\alpha\left(\frac{b}{d} - I - Q\right)\right]^2} - \frac{(\theta + d)}{I} < 0$$

Thus there is no limit cycle i.e. no periodic solutions exist in the region. Hence by Poincare-Bendixson theory, endemic equilibrium E^* is globally asymptotically stable in the region $\Omega - \{(S, I, Q) : I = 0\}$ for the limit system (4.2) and hence for the original system (2.1).

VI. NUMERICAL SIMULATIONS AND CONCLUDING REMARKS

In this paper we have carried out the global analysis of an SIQS model and observe that the basic quarantine number R_q plays an important role to control the disease. Our main results show that if $R_q < 1$, the disease-free equilibrium is globally stable and if $R_q > 1$, then the endemic equilibrium exists and is globally stable. Beside this analytical study, we provide some numerical solutions as under:

Disease-free equilibrium: Choosing the parameters as under

$b = 8, d = 0.1, \lambda = 0.01, \gamma = 0.4, \alpha = 1, \phi = 0.3, d_1 = 0.2, \theta = 0.15, (S(0), I(0), Q(0)) = (50, 20, 10)$ gives $R_q < 1$ and in this case $S(t)$ approaches to its steady state value while $I(t)$ and $Q(t)$ approaches to zero as time goes to infinity, the disease disappears and dies out (fig.5.1).

Endemic equilibrium: Choosing the parameters as under

$b = 10, d = 0.1, \lambda = 0.6, \gamma = 0.3, \alpha = 0.5, \phi = 0.4, d_1 = 0, \theta = 0.1, (S(0), I(0), Q(0)) = (50, 20, 10)$, gives $R_q > 1$ and in this case all the components $S(t)$, $I(t)$ and $Q(t)$ approaches to their steady state values as time goes to infinity, the disease becomes endemic (fig.5.2).

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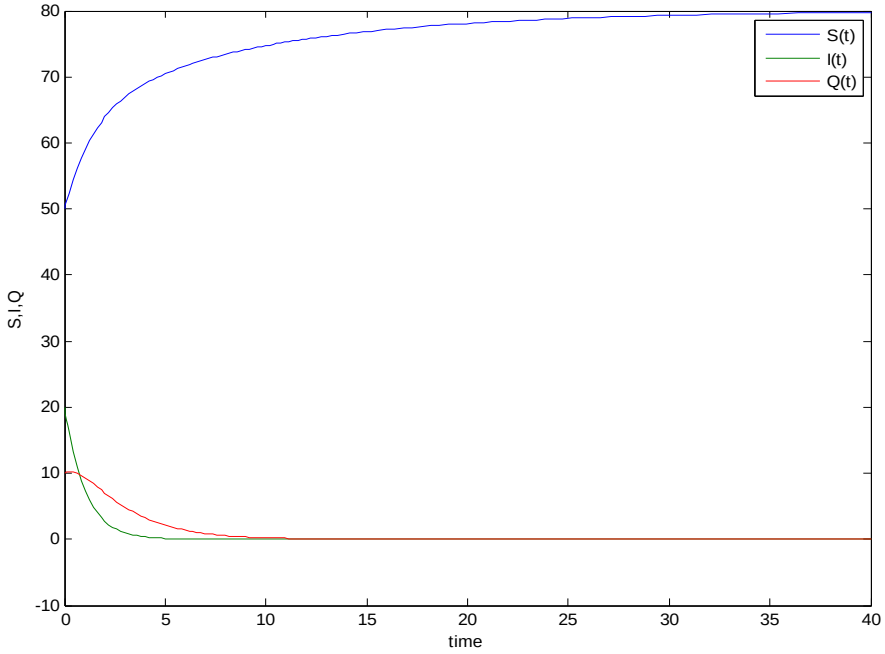

Figure 5.1: This figure shows that the disease dies out

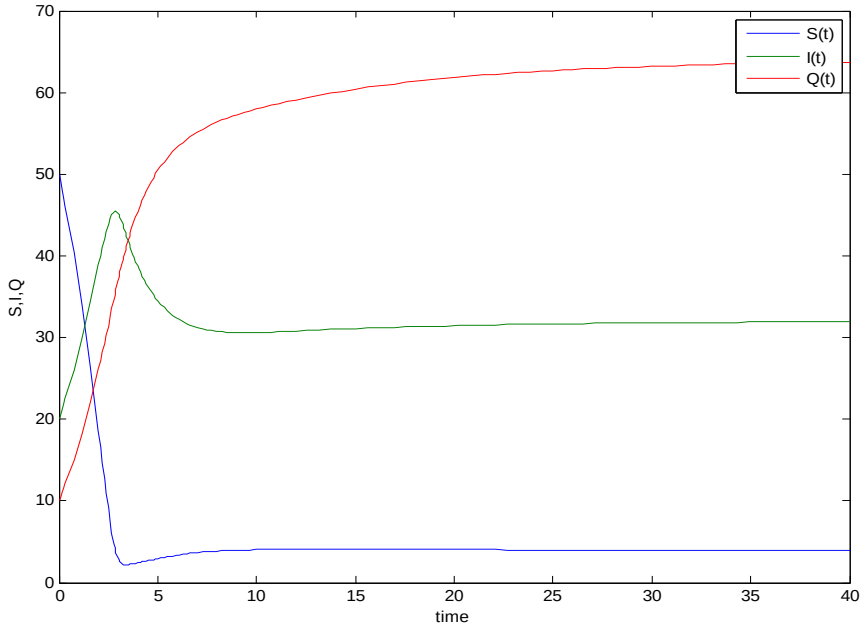

Figure 5.2: This figure shows that the disease becomes endemic

The quarantine process is one of the methods for reducing the average infectious period by isolating some infectives, so that they do not transmit the infection. In above SIQS model with quarantine, we can see that the

effective infectious period $1/(\gamma+\phi+d+d_1)$ and R_q decreases as the quarantine rate constant ϕ increases. Since the mean residence time in the quarantine class Q is $1/\theta$ and the expression for R_q is independent from parameter θ, it shows that the assumption of quarantine models that the people in the quarantine class Q do not infect others and people are not infectious when they move out of the quarantine class, is acceptable. Also it is clear from the expressions of R_q that it depends on α and is decreases as α increases, it indicates that the spread of diseases decreases as the psychological protective measures for the infective increases.

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