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On weekly-$\mathfrak{I}$-regular and Weekly- $\mathfrak{I}$-normal spaces

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ABSTRACT

In this paper *we will introduce weekly-$\mathfrak{I}$-regular and weekly-$\mathfrak{I}$-normal spaces and give some characterizations of these spaces.*

Key Words and phrases: **weekly-$\mathfrak{I}$-regular, weekly-$\mathfrak{I}$-normal.**

2000 **MSC**: 54D15, 54C10.

I.INTRODUCTION

It is very well known concept about Higher Separation Axioms Regularity and Normality in literature. On the other hand Higher Separation axioms with respect to an ideal and various properties and characterizations were also discussed by many authors. Ideals in topological spaces were introduced by Kuratowski[2] and further studied by Vaidyanathaswamy[4]. Corresponding to an ideal a new topology $\tau^*(\mathfrak{I}, \tau)$ called the *-topology was given which is generally finer than the original topology having the kuratowski closure operator $cl^*(A) = A \cup A^*(\mathfrak{I},\tau)$[5], where $A^*(\mathfrak{I},\tau) = \{x \in X : U \cap A \notin \mathfrak{I}$ for every open subset U of x in X called a local function of A with respect to $\mathfrak{I}$ and τ. We will write τ^* for $\tau^*(\mathfrak{I}, \tau)$.

The following section contains some definitions and results that will be used in our further sections.

Definition 1.1.[2]: Let (X, τ) be a topological space. An ideal $\mathfrak{I}$ on X is a collection of non-empty subsets of X such that (a) $\phi \in \mathfrak{I}$ (b) $A \in \mathfrak{I}$ and $B \in \mathfrak{I}$ implies $A \cup B \in \mathfrak{I}$ (c) $B \in \mathfrak{I}$ and $A \subset B$ implies $A \in \mathfrak{I}$.

Definition 1.2.[3]: An ideal topological space $(X, \tau, \mathfrak{I})$ is said to be $\mathfrak{I}$-regular if for any closed set F in X and a point $a \notin F$, there exist open sets U and V in X such that $U \cap V = \phi$ and $a \in U$, $F-V \in \mathfrak{I}$.

Definition 1.3.[3] : Let $(X,\tau,\mathfrak{I})$ be an ideal space then X is said to be $\mathfrak{I}$-normal if for any pair of disjoint closed sets A and B of X, there exist disjoint open sets G and H in X such that $A-G \in \mathfrak{I}$ and $B-H \in \mathfrak{I}$.

Lemma 1.4.[1]: Let $(X,\tau,\mathfrak{I})$ be an ideal space and A be any subset of X. Then the following holds:

(a) $A^* = cl(A^*) \subset cl(A)$ and so A^* is a closed subset of cl(A).
(b) τ is compatible to $\mathfrak{I}$ i.e. $\tau \sim \mathfrak{I}$ iff $A - A^* \in \mathfrak{I}$.

. **Lemma 1.5.[1] :** Let $(X,\tau,\mathfrak{I})$ be an ideal space and Y be subset of X. Then

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$\mathfrak{I}_Y$ = {I∩Y | I ∈   } is an ideal on Y

II.RESULTS

Lemma 2.1: Let $(X,\tau,\mathfrak{I})$ be an ideal and G and H are two open subsets of X such that $G \cap H \in \mathfrak{I}$. Then prove that $G^* \cap H = \phi$.

Proof: On contrary, let $G^* \cap H \neq \phi$. Let $a \in G^* \cap H$. This implies $a \in G^*$ and $a \in H$.

But $a \in G^*$ implies that if $a \in U$ and U is open set then $U \cap G \notin \mathfrak{I}$. Further, $a \in H$ i.e. H is open set containing a.

Therefore, H is open set containing a and $G \cap H \in \mathfrak{I}$, a contradiction.

Hence $G^* \cap H = \phi$.

Definition 2.2: Let $(X,\tau,\mathfrak{I})$ be an ideal space then X is said to be weekly-$\mathfrak{I}$-regular if for any closed set F in X and a point $a \notin F$, there exist open sets U and V in X such that $U \cap V \in \mathfrak{I}$ and $a \in U$, $F-V \in \mathfrak{I}$.

Remark 2.3: Every $\mathfrak{I}$-regular space is weekly-$\mathfrak{I}$-regular but the converse is not true as can be seen from the following example:

Example 2.4: Let X={a,b,c}, τ={ϕ, {a}, {a,b}, X} and $\mathfrak{I}$={ ϕ,{a},{c},{a,c}} and so τ^*={ ϕ, {a}, {b},{a,b},{b,c},X}. Then no two open sets in X are disjoint. Therefore, X is not $\mathfrak{I}$-regular. But it can be easily seen that X is weekly-$\mathfrak{I}$-regular, since {b,c} is closed set and a ∉ {b,c} implies that there exist open sets {a} and {a,b} such that {a}∩{a,b} ∈ $\mathfrak{I}$ and a ∈ {a}, {b,c}-{a,b} = {c} ∈ $\mathfrak{I}$. Also for the closed set {c} and {a}∉{c} there exist open sets {a} and {a,b} and for b∉{c}, there exist open sets {a,b} and {a}.

Theorem 2.5: If $(X,\tau,\mathfrak{I})$ is an $\mathfrak{I}$ –regular space and $Y \subset X$, then prove that $(Y,\tau_Y, \mathfrak{I}_Y)$ is $\mathfrak{I}$ –regular.

Proof: Let F be an closed subset of Y and $a \in Y$ be any element such that $a \notin F$. By definition of subspace topology, $F=L \cap Y$ where L is closed in X and $a \notin L$. But X is $\mathfrak{I}$ –regular implies that there exist open sets U and V in X such that $U \cap V \in \mathfrak{I}$ and $a \in U$, $L-V \in \mathfrak{I}$. Therefore, $U \cap Y$ and $V \cap Y$ are open in Y such that $a \in U \cap Y$, $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y \in \mathfrak{I}_Y$. Also $L-V \in \mathfrak{I}$ implies that L-V = I for some $I \in \mathfrak{I}$ and so $L \subset V \cup I$ and so $L \cap Y \subset (V \cup I) \cap Y = (V \cap Y) \cup (I \cap Y)$. Therefore, $F-(V \cap Y) \subset I \cap Y \in \mathfrak{I}_Y$ implies that $F-(V \cap Y) \in \mathfrak{I}_Y$. Hence $(Y,\tau_Y, \mathfrak{I}_Y)$ is $\mathfrak{I}$-regular.

Theorem 2.6: Let $(X,\tau,\mathfrak{I})$ be an ideal space and $\tau \sim \mathfrak{I}$. Then the following are equivalent:

a) X is weekly $\mathfrak{I}$ –regular.
b) For each $x \in X$ and open set G containing x, there is an open set H containing x such that $H^*-G \in \mathfrak{I}$.
c) For each $x \in X$ and closed set F not containing x, there is an open set H containing x such that

$H^* \cap F \in \mathfrak{I}$.

Proof: (a)⇒(b): Let $x \in X$ and G be any open set containing x, then X-G is closed set not containing x. Therefore, (a) implies that there exist open sets H and K such that $x \in H$, $(X-G)-K \in \mathfrak{I}$ and $H \cap K \in \mathfrak{I}$ and so by Lemma 2.1 , $H^* \cap K = \phi$. Therefore, $H^* \subset K^c$ implies that $(X-G) \cap H^* \in \mathfrak{I}$ and so $H^*-G \in \mathfrak{I}$. Hence (b) holds.

(b)⇒(c): Let F be any closed set not containing x, then X-F is open set containing x. Therefore, (b) implies that there exist open set H containing x such that $H^*-(X-F) \in \mathfrak{I}$ and so $H^* \cap F \in \mathfrak{I}$. Hence (c) holds.

(c) ⇒(a): Let F be any closed set not containing x, then by (c) there is an open set H containing x such that $H^* \cap F \in \mathfrak{I}$. Therefore, H and $X-H^*$ are open sets in X such that $x \in H$, $F-(X-H^*) \in \mathfrak{I}$. Also $H \cap (X-H^*) = H-H^* \in \mathfrak{I}$ by Lemma 1.4. Hence (a) holds.

Definition 2.7: Let $(X,\tau,\mathfrak{I})$ be an ideal space then X is said to be weekly-$\mathfrak{I}$-normal if for any pair of disjoint closed sets A and B of X, there exist open sets G and H in X such that $G \cap H \in \mathfrak{I}$ and $A-G \in \mathfrak{I}$, $B-H \in \mathfrak{I}$.

Remark 2.8: It can be seen easily that every $\mathfrak{I}$-normal space is weekly-$\mathfrak{I}$-normal but the converse is not true as can be seen from the following example.

Example 2.9: Let X be any infinite set with cofinite topology and $\tau = \wp(X) \equiv$ Collection of all subsets of X. Then it can be seen easily that X is weekly-$\mathfrak{I}$-normal, since every subset of X belongs to $\mathfrak{I}$. But it is not

$\mathfrak{I}$-normal space, since there does not exist any disjoint open sets in X.

Theorem 2.10: If $(X,\tau,\mathfrak{I})$ is an weekly $\mathfrak{I}$–normal space and $Y \subset X$ is closed subset of X, then prove that

$(Y,\tau_Y, \mathfrak{I}_Y)$ is $\mathfrak{I}$ –normal.

Proof: Let F and K be two disjoint closed subsets of Y. Then by definition of subspace topology, Y is closed in X implies F and K are closed in X. Now X is weekly $\mathfrak{I}$–normal implies that there exist open sets G and H in X such that $G \cap H \in \mathfrak{I}$ and $F-G \in \mathfrak{I}$, $K-H \in \mathfrak{I}$. Therefore, $G \cap Y$ and $H \cap Y$ are open in Y such that $(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y \in \mathfrak{I}_Y$. Also $F-G \in \mathfrak{I}$ implies that $F-G = I$ for some $I \in \mathfrak{I}$ and so $F \subset G \cup I$ and so $F \cap Y \subset (G \cup I) \cap Y = (G \cap Y) \cup (I \cap Y)$. Therefore, $F-(G \cap Y) \subset I \cap Y \in \mathfrak{I}_Y$ and similarly $K-(H \cap Y) \in \mathfrak{I}_Y$. Hence $(Y,\tau_Y,\mathfrak{I}_Y)$ is

$\mathfrak{I}$-normal.

Theorem 2.11: Let $(X,\tau,\mathfrak{I})$ be an ideal space. Then the following conditions are equivalent:

a) X is weekly $\mathfrak{I}$–normal.
b) For every closed set K and open set U such that $K \subset U$, there is an open set H such that

K-H $\in \mathfrak{T}$ and H^*-U $\in \mathfrak{T}$.

c) For each pair of disjoint closed sets F and K, there is an open set H such that F-H $\in \mathfrak{T}$ and $H^* \cap K \in \mathfrak{T}$.

Proof: (a)⇒(b): Let K be any closed set and U be any open set containing K, then $K \cap (X-U)=\phi$ i.e. K and X-U are a pair of disjoint closed sets. Therefore, by (a) there exist open sets G and H such that K-G $\in \mathfrak{T}$,

(X-U)-H $\in \mathfrak{T}$ and $G \cap H \in \mathfrak{T}$. Thus by Lemma 2.1, $G^* \cap H=\phi$ implies that $G^* \subset H^C$ and so $(X-U) \cap G^* \in \mathfrak{T}$. Hence there exist open set G such that K-G $\in \mathfrak{T}$ and G^*-U $\in \mathfrak{T}$.

(b)⇒(c): Let F and K are a pair of disjoint closed sets, then $F \subset X-K$, where X-K is open.

Therefore, by (b) there exist an open set H such that F-H $\in \mathfrak{T}$ and H^*-(X-K) $\in \mathfrak{T}$ and so F-H $\in \mathfrak{T}$ and $H^* \cap K \in \mathfrak{T}$.

(c)⇒(a): Let F and K are a pair of disjoint closed sets in X, then by (c) there exist an open set H such that F-H $\in \mathfrak{T}$ and $H^* \cap K \in \mathfrak{T}$. Therefore, H and X-$H^*$ are open sets in X such that F-H $\in \mathfrak{T}$, K-(X-H^*) $\in \mathfrak{T}$ and

$H \cap (X-H^*)=H-H^* \in \mathfrak{T}$ by Lemma 1.4. Hence (a) holds.

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