On θ_I Kernel of a set

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ABSTRACT

In this paper *we will introduce* θ_I *kernel of a set and give its characterizations. Also Examples are given throughout the paper.*

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I.INTRODUCTION

In [2], Csàszàr, introduced S_1 and S_2 spaces and discussed some properties of these spaces and in [3], Janković gave various characterizations of S_1 and S_2 spaces using the θ-closure of a set and θ-Kernel of a set. On the other hand separation axioms with respect to an ideal and various properties and characterizations were also discussed by many authors. Ideals in topological spaces were introduced by Kuratowski[4] and further studied by Vaidyanathaswamy[5]. Corresponding to an ideal a new topology $\tau^*(\mathfrak{I}, \tau)$ called the *-topology was given which is generally finer than the original topology having the kuratowski closure operator $cl^*(A) = A \cup A^*(\mathfrak{I},\tau)$[6], where $A^*(\mathfrak{I},\tau) = \{x \in X : U \cap A \notin \mathfrak{I}$ for every open subset U of x in X called a local function of A with respect to $\mathfrak{I}$ and τ. We will write τ^* for $\tau^*(\mathfrak{I}, \tau)$.

The following section contains some definitions and results that will be used in our further sections.

Definition 1.1.[4]: Let (X, τ) be a topological space. An ideal $\mathfrak{I}$ on X is a collection of non-empty subsets of X such that (a) $\phi \in \mathfrak{I}$ (b) $A \in \mathfrak{I}$ and $B \in \mathfrak{I}$ implies $A \cup B \in \mathfrak{I}$ (c) $B \in \mathfrak{I}$ and $A \subset B$ implies $A \in \mathfrak{I}$.

Definition 1.2.[2]: A topological space (X, τ) is said to be S_1 space if for every pair of distinct points x and y, whenever one of them has a open set not containing the other then the other also has a open set not containing the other.

Definition 1.3.[1]: Let $(X,\tau,\mathfrak{I})$ be an ideal space. Then for any subset A of X, a point x is said to be in the θ_I closure of A if for every open subset U of x in X, $cl^*(U) \cap A \neq \phi$. The collection of all such points is denoted by $cl_{\theta_I}(A)$. Also A is said to be θ_I closed if $cl_{\theta_I}(A)$ =A.

Definition 1.4.[3]: Let (X,τ) be a topological space and $x \in X$ be any element. Then

a) $Ker\{x\} = \bigcap\{G : G \in \tau(x)\}$, where τ(x) denotes the collection of all open subsets of x i.e. Ker{x} = { y ∊ X | cl{y} ∩ {x} ≠ ϕ}.

(b) $Ker_{\theta}(A)$ = { x ∊ X | $cl_{\theta}(x) \cap A \neq \phi$}

II.RESULTS

We will begin by defining the θ-Kernel of a set.

Definition 2.1: Let (X,τ,$\mathfrak{I}$) be an ideal space and A be any subset of X. Then

$Ker_{\theta_I}(A)$ = { x ∊ X | $cl_{\theta_I}(x) \cap A \neq \phi$}.

Remark 2.2: Since, we know that cl*(A) ⊂ cl(A) for any subset A of X. So, by definition it is obvious that $cl_{\theta_I}(A) \subset cl_{\theta}(A)$ for any subset A of X. Hence it follows that $Ker_{\theta_I}(A) \subset Ker_{\theta}(A)$ for any subset A of X. But the following Example shows that the converse is not true.

Example 2.3: Let X={a,b,c}, τ={ϕ, {b}, {a,b}, X} and $\mathfrak{I}$={ ϕ,{a},{b},{a,b}} and so τ^*=℘(X). Then

$Ker_{\theta_I}(b)$ ={b} and $Ker_{\theta}(b)$ = {a,b,c}.

Hence $Ker_{\theta}(b) \not\subset Ker_{\theta_I}(b)$.

Theorem 2.4: Let (X,τ,$\mathfrak{I}$) be an ideal space. Then prove that the following holds:

(a) For each A ⊂ X, A ⊂ Ker(A) ⊂ $Ker_{\theta_I}(A)$
(b) If A ⊂ B ⊂ X then $Ker_{\theta_I}(A) \subset Ker_{\theta_I}(B)$
(c) If A , B ⊂ X then $Ker_{\theta_I}(A \cup B) = Ker_{\theta_I}(A) \cup Ker_{\theta_I}(B)$
(d) If X is S_1 space then prove that $Ker_{\theta_I}(A) \subset cl_{\theta_I}(A)$.
(e) If X is S_1 space and A is any compact subset of X then $cl_{\theta_I}(A) \subset Ker_{\theta_I}(A)$.

Proof: (a) Let A be any subset of X. Then ∀ x ∊ A, x ∊ cl{x} implies cl{x} ∩ A ≠ ϕ.

Therefore, A ⊂ ker(A). Also cl{x} ⊂ $cl_{\theta_I}(x)$ implies that Ker(A) ⊂ $Ker_{\theta_I}(A)$.

Hence A ⊂ Ker(A) ⊂ $Ker_{\theta_I}(A)$.

(b) Let A,B be two subsets of X such that A ⊂ B. Then $cl_{\theta_I}(A) \subset cl_{\theta_I}(B)$ implies that $Ker_{\theta_I}(A) \subset Ker_{\theta_I}(B)$. Hence (b) holds.

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(c) Let A,B be two subsets of X. Then A ⊂ A∪B and B ⊂ A∪B, so by (b) $Ker_{\theta_I}(A) \subset Ker_{\theta_I}(A\cup B)$ and $Ker_{\theta_I}(B) \subset Ker_{\theta_I}(A\cup B)$. Therefore, we have $Ker_{\theta_I}(A) \cup Ker_{\theta_I}(B) \subset Ker_{\theta_I}(A\cup B)$. Conversely, let x ϵ $Ker_{\theta_I}(A\cup B)$ implies that $cl_{\theta_I}(x) \cap (A\cup B) \neq \phi$. So $(cl_{\theta_I}(x) \cap A) \cup (cl_{\theta_I}(x) \cap B) \neq \phi$. This implies that either $cl_{\theta_I}(x) \cap A \neq \phi$ or $cl_{\theta_I}(x) \cap B \neq \phi$ and so either x ϵ $Ker_{\theta_I}(A)$ or x ϵ $Ker_{\theta_I}(B)$.

Therefore, x ϵ $Ker_{\theta_I}(A) \cup Ker_{\theta_I}(B)$. Hence $Ker_{\theta_I}(A) \cup Ker_{\theta_I}(B) = Ker_{\theta_I}(A\cup B)$. Hence (c) holds.

(d) Let X be S_1 space and A be any subset of X. Let y ∉ $cl_{\theta_I}(A)$. We have to prove that y ∉ $Ker_{\theta_I}(A)$ i.e. we have to prove that $cl_{\theta_I}(y) \cap A = \phi$. Let z ϵ A then we have to prove that z ∉ $cl_{\theta_I}(y)$. Now y ∉ $cl_{\theta_I}(A)$ implies that there exist open set U_y containing y such that $cl^*(U_y) \cap A = \phi$. Further, z ϵ A and $cl^*(U_y) \cap A = \phi$ implies that z ∉ $cl^*(U_y)$ and so z ∉ U_y and z ∉ U_y^*. Now, z ∉ U_y implies that y has a open set U_y not containing z and so X is S_1 implies that z has a open set say U_z not containing y. And z ∉ U_y^* implies that there exist open set V_z containing z such that $V_z \cap U_y \in \mathfrak{I}$. Consider $H_z = U_z \cap V_z$. Then H_z is open set containing z but not y. Also $H_z \cap U_y \in \mathfrak{I}$. Therefore, y ∉ H_z and y ∉ H_z^* implies that y ∉ $cl^*(H_z)$ i.e.

$cl^*(H_z) \cap \{y\} = \phi$ and so z ∉ $cl_{\theta_I}(y)$. Hence z ∉ $Ker_{\theta_I}(A)$.

(e): Let A be any compact subset of X and X is S_1. We have to prove that $cl_{\theta_I}(A) \subset Ker_{\theta_I}(A)$. Let y ∉ $Ker_{\theta_I}(A)$. Then $cl_{\theta_I}(y) \cap A = \phi$. This implies that ∀ z ϵ A, z ∉ $cl_{\theta_I}(y)$ and so ∀ z ϵ A there exist U_z such that $cl^*(U_z) \cap \{y\} = \phi$. Therefore, y ∉ U_z and y ∉ U_z^*. Hence ∀ z ϵ A, there exist open set V_z containing y such that $V_z \cap U_z \in \mathfrak{I}$. Further, z has a open set U_z not containing y and so X is S_1 implies that y has a open set say G_z not containing z. Consider $H_z = G_z \cap V_z$. Then ∀ z ϵ A H_z is open set containing y but not z and U_z is open set containing z such that $H_z \cap U_z \in \mathfrak{I}$.Further, $A \subset \bigcup_{z\in A} U_z$ and A is compact implies that there exist finite subset A_0 of A such that $A \subset \bigcup_{z\in A_0} U_z$. Let $U = \bigcup_{z\in A_0} U_z$ and $V = \bigcap_{z\in A_0} H_z$ then

$U \cap V \in \mathfrak{I}$. Now, since ∀ z ϵ A, z ∉ H_z. This implies that $V \cap A = \phi$. Also $U \cap V \in \mathfrak{I}$ implies that $V^* \cap U = \phi$ and so A ⊂ U implies that $V^* \cap A = \phi$. Hence $cl^*(V) \cap A = \phi$ implies that y ∉ $cl_{\theta_I}(A)$.

Hence $cl_{\theta_I}(A) \subset Ker_{\theta_I}(A)$.

If X is S_1 and A be compact subset of X then $cl_{\theta_I}(A) = Ker_{\theta_I}(A)$.

The following Example shows that the result is not true if the space is not S_1.

Example 2.5: Let X={a,b,c}, τ = {ϕ, {a}, {b}, {a,b}, X} and $\mathfrak{I}$ = { ϕ,{a},{b},{a,b}} and so $\tau^* = \wp(X)$. Then it can be seen easily that X is not S_1. Since, 'a' has a open set {a} not containing 'c' but 'c' does not have any open set not containing 'a'. Also $cl_{\theta_I}(a)$ = {a,c} and $Ker_{\theta_I}(a)$ = {a} and so $cl_{\theta_I}(a) \not\subset Ker_{\theta_I}(a)$. And $cl_{\theta_I}(c)$ = {c} and $Ker_{\theta_I}(c)$ = {a,b,c} and so $Ker_{\theta_I}(c) \not\subset cl_{\theta_I}(c)$.

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