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Characterization of regular spaces via ideal

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ABSTRACT

In this paper *we will give some characterizations of* $\mathfrak{I}$-regular and weekly-$\mathfrak{I}$-regular *spaces. Also Examples are given throughout the paper.*

Key Words and phrases: ***S_2 mod $\mathfrak{I}$,*** **$\mathfrak{I}$-regular, weekly-$\mathfrak{I}$-regular.**

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I.INTRODUCTION

In [1], Csàszàr, introduced regular spaces and discussed some properties of these spaces and give its characterization in terms of Rim compact spaces. On the other hand separation axioms with respect to an ideal and various properties and characterizations were also discussed by many authors. Ideals in topological spaces were introduced by Kuratowski[4] and further studied by Vaidyanathaswamy[5]. Corresponding to an ideal a new topology $\tau^*(\mathfrak{I}, \tau)$ called the *-topology was given which is generally finer than the original topology having the kuratowski closure operator $cl^*(A) = A \cup A^*(\mathfrak{I},\tau)$[6], where $A^*(\mathfrak{I},\tau) = \{x \in X : U \cap A \notin \mathfrak{I}$ for every open subset U of x in X called a local function of A with respect to $\mathfrak{I}$ and τ. We will write τ^* for $\tau^*(\mathfrak{I}, \tau)$.
The following section contains some definitions and results that will be used in our further sections.

Definition 1.1.[4]: Let (X, τ) be a topological space. An ideal $\mathfrak{I}$ on X is a collection of non-empty subsets of X such that (a) $\phi \in \mathfrak{I}$ (b) $A \in \mathfrak{I}$ and $B \in \mathfrak{I}$ implies $A \cup B \in \mathfrak{I}$ (c) $B \in \mathfrak{I}$ and $A \subset B$ implies $A \in \mathfrak{I}$.

Definition 1.2.[1]: A topological space (X, τ) is said to be S_2 space if for every pair of distinct points x and y, whenever one of them has a open set not containing the other then there exist disjoint open subsets containing them.

Definition 1.3.: An ideal space $(X,\tau, \mathfrak{I})$ is said to be **S_2 mod $\mathfrak{I}$** if for every pair of distinct points x and y in X, whenever x has a τ- open subset not containing y, there exist open nhds. U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathfrak{I}$.

Definition 1.4.[3]: An ideal topological space $(X, \tau, \mathfrak{I})$ is said to be $\mathfrak{I}$-regular if for any closed set F in X and a point $a \notin F$, there exist open sets U and V in X such that $U \cap V = \phi$ and $a \in U$, $F - V \in \mathfrak{I}$.

Definition 1.5.[2]: Let $(X,\tau,\mathfrak{I})$ be an ideal space and A be any subset of X. Then the following holds:

(a) $A^* = cl(A^*) \subset cl(A)$ and so A^* is a closed subset of cl(A).

(b) τ is compatible to $\mathfrak{I}$ i.e. $\tau \sim \mathfrak{I}$ iff $A - A^* \in \mathfrak{I}$.

II.RESULTS

Definition 2.1 : Let $(X,\tau,\mathfrak{I})$ be an ideal space. Then X is said to be Rim-$\mathfrak{I}$-compact if U be any open set containing x then there exist basic open set V containing x such that $x \in V \subset U$ and Fr(V) is $\mathfrak{I}$-compact.

Theorem 2.2 : Prove that every Rim-$\mathfrak{I}$-compact S_2 space is $\mathfrak{I}$-regular.

Proof: Let $(X,\tau,\mathfrak{I})$ be an ideal space and x be S_2 and let U be any open set containing x i.e. $x \in U$. Then we have to find open set G such that

$x \in G \subset cl(G)$ and $Cl(G)\text{-}U \in \mathfrak{I}$.

Now X is Rim-$\mathfrak{I}$-compact implies that there exist basic open set V containing x such that

$x \in V \subset U$ and Fr(V) is $\mathfrak{I}$-compact.

Further, $\forall\, y \in Fr(V)$, $y \notin V$, since $y \in Fr(V)$ implies that $y \in cl(V)$ but $y \notin V$.

Therefore, x has a open set V not containing y.

Now, X is S_2 space implies that there exist open sets U_y and V_y containing x and y respectively such that

$U_y \cap V_y = \phi$ and so $Fr(V) \subset \bigcup_{y\in Fr(V)} V_y$. Therefore, Fr(V) is $\mathfrak{I}$-compact implies that there exist finite subset A of Fr(V) such that $Fr(V) - \bigcup_{y\in A} V_y \in \mathfrak{I}$.

Let $G = \bigcup_{y\in A} V_y$ and $H = \bigcap_{y\in A} U_y$. Then $Fr(V)\text{-}G \in \mathfrak{I}$ and H is open set containing x. Also $G \cap H = \phi$

and so $G \cap cl(H) = \phi$ and so $cl(H) \subset G^C$. Therefore, $Fr(V) - G \in \mathfrak{I}$ implies that $Fr(V) \cap cl(H) \in \mathfrak{I}$ and so

$(cl(V)\text{-}int(V))\cap cl(H) = cl(V)\cap cl(H)\text{-}int(V) \in \mathfrak{I}$ implies $cl(V\cap H) - int(V) \in \mathfrak{I}$ and so $cl(V\cap H) - V \in \mathfrak{I}$, since V is open and $cl(V\cap H) \subset cl(V) \cap cl(H)$.

Now consider $W = V \cap H$ is the open set containing x. So, $cl(W) - V \in \mathfrak{I}$ and so $cl(W) - U \in \mathfrak{I}$.

Hence W is the open set containing x such that $cl(W) - U \in \mathfrak{I}$.

Hence X is $\mathfrak{I}$-regular.

The following Example shows that the result is not true if we replace S_2 space by S_2 mod $\mathfrak{I}$ space.

Example 2.3 : Let X= N ≡ the set of natural numbers with cofinite topology i.e. $\tau = \{ G \subset X \mid X\text{-}G \text{ is finite}\} \cup \phi$.

and $\mathfrak{I} = \wp(X) \equiv$ Collection of all subsets of X.

Then X is $\mathfrak{I}$-compact obviously since every subset of X is a member of $\mathfrak{I}$ and so X is Rim-$\mathfrak{I}$-compact. Also X is S_2 mod $\mathfrak{I}$. But X is not $\mathfrak{I}$-regular. Since there does not exist any disjoint open sets.

Before, we start our next result, firstly we define weekly-$\mathfrak{I}$-regular spaces.

Definition 2.4 : Let $(X,\tau,\mathfrak{I})$ be an ideal space then X is said to be weekly-$\mathfrak{I}$-regular if for any closed set F in X and a point $a \notin F$, there exist open sets U and V in X such that $U \cap V \in \mathfrak{I}$ and $a \in U$, $F\text{-}V \in \mathfrak{I}$.

Theorem 2.5: Prove that every Rim-$\mathfrak{I}$-compact S_2 mod $\mathfrak{I}$ space is weekly-$\mathfrak{I}$-regular.

Proof: Let $(X,\tau,\mathfrak{I})$ be an ideal space and let us assume that $\tau \sim \mathfrak{I}$ and let U be any open set containing x i.e. $x \in U$. Then we have to find open set G such that $x \in G \subset cl(G)$ and $G^*\text{-}U \in \mathfrak{I}$.

Now X is Rim-$\mathfrak{I}$-compact implies that there exist basic open set V containing x such that

$x \in V \subset U$ and Fr(V) is $\mathfrak{I}$-compact.

Further, $\forall\, y \in Fr(V)$, $y \notin V$, since $y \in Fr(V)$ implies that $y \in cl(V)$ but $y \notin V$.

Therefore, x has a open set V not containing y.

Now, X is S_2 mod $\mathfrak{I}$ space implies that there exist open sets U_y and V_y containing x and y respectively such that $U_y \cap V_y \in \mathfrak{I}$ and so $Fr(V) \subset \bigcup_{y\in Fr(V)} V_y$. Therefore, Fr(V) is $\mathfrak{I}$-compact implies that there exist finite subset A of Fr(V) such that $Fr(V) - \bigcup_{y\in A} V_y \in \mathfrak{I}$.

Let $G = \bigcup_{y\in A} V_y$ and $H = \bigcap_{y\in A} U_y$. Then $Fr(V)\text{-}G \in \mathfrak{I}$ and H is open set containing x. Also $G \cap H \in \mathfrak{I}$ and so $G \cap H^* = \phi$ and so $H^* \subset G^C$. Therefore, $Fr(V) - G \in \mathfrak{I}$ implies that $Fr(V) \cap H^* \in \mathfrak{I}$ and so

$(cl(V)\text{-}int(V)) \cap H^* = cl(V) \cap H^*\text{-}int(V) \in \mathfrak{I}$ implies $V^* \cap H^* - int(V) \in \mathfrak{I}$ and so $(V \cap H)^* - V \in \mathfrak{I}$, since V is open and $(V \cap H)^* \subset V^* \cap H^*$.

Now consider $W = V \cap H$ is the open set containing x. So, $W^* - V \in \mathfrak{I}$ and so $W^* - U \in \mathfrak{I}$.

Hence W is the open set containing x such that $W^* - U \in \mathfrak{I}$.

Hence X is weekly-$\mathfrak{I}$-regular.

Since we know that every regular space is S_2 space but the following example shows that that the

$\mathfrak{I}$-regular space is not S_2 mod $\mathfrak{I}$.

Example 2.6 : Let X={a,b,c} with τ = { ϕ,{a},{b},{a,b}} and $\mathfrak{I}$={ϕ,{c}}. Then X is $\mathfrak{I}$-regular. Since closed sets are {b,c},{a,c} and {c}. Now for a ∉ {b,c}, there exist open sets {a} and {b} such that a ϵ {a} and {b,c}-{b} ϵ $\mathfrak{I}$ and {a} ⋂{b} = ϕ. And for b∉ {a,c}, there exist open sets {b} and {a} and for a∉ {c}, there exist open sets {a} and {b} and for b∉ {c}, there exist open sets {b} and {a}. Hence X is $\mathfrak{I}$-regular.

But X is not S_2 mod $\mathfrak{I}$. Since 'a' has a open set {a} not containing 'c' But there does not exist any open sets U and V such that a ϵ U and b ϵ V and U⋂V ϵ $\mathfrak{I}$.

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