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Skew signless laplacian energy of Oriented graphs

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ABSTRACT

Let D be a oriented graph with skew signless Laplacian Matrix $SSL(D)$. The skew signless Laplacian energy of oriented graph D is defined as the sum of the norms of all eigen values of $SSL(D)$. Two oriented graphs are said to be skew signless Laplacian equienergetic if their skew signless Laplacian energies are equal. In this paper we obtain the skew signless Laplacian energy $SSL(D)$ of complete oriented bipartite graph $K_{n,2}$ and $K_{n,m}$ and show that skew signless Laplacian energies of these complete oriented graphs are always an even number.

MATHEMATICS SUBJECT CLASSIFICATION: *05C50, 05C30.*

KEY WORDS AND PHRASES: Signless Laplacian spectra, Skew signless Laplacian energy of digraph.

I. INTRODUCTION

Let D be a oriented graph with n vertices $v_1, v_2, ...v_n$ and m arcs. Let $d_i^+ = d^+(v_i), d^i = d^-(v_i)$ and $d_i = d_i^+ + d_i^-$, $i = 1, 2, \ldots, n$ be the outdegree, indegree and degree of vertices of D, respectively. The out-adjacency matrix $A^+(D) = (a_{ij})$ of a digraph D is the $n \times n$ matrix, where $a_{ij} = 1$, if (v_i, v_j) is an arc and $a_{ij} = 0$, otherwise. The in adjacency matrix $A^-(D) = (a_{ij})$ of a digraph D is the $n \times n$ matrix, where $a_{ij} = 1$, if (v_j, v_i) is an arc and $a_{ij} = 0$. It is clear that $A^-(D) = (A^+(D))^t$.

The skew adjacency matrix S(D) = (s_{ij}) of a digraph D is the $n \times n$ matrix, where (s_{ij}) = 1, if there is an arc from v_i to v_j, (s_{ij}) = -1, if there is an arc from v_j to v_i and (s_{ij}) = 0, otherwise.

It is clear that S(D) is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. The energy of the matrix S(D) was considered in [1], and is defined as

$$E_s(D) = \sum_{i=1}^{n} |\zeta_i|,$$

where $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the eigen values of S(D). This energy of a digraph D is called the skew energy by Adiga et al. [1]. For recent developments in the theory of skew energy, see the survey [10].

Let $D^{+}(G) =$ diag($d_1^{+}, d_2^{+}, \ldots, d_n^{+}$), $D^{-}(G) =$ diag($d_1^{-}, d_2^{-}, \ldots, d_n^{-}$) and D(G) = diag($d_1, d_2, \ldots, d_n$) be the diagonal of vertex out degrees, vertex in degrees and vertex degrees of D respectively.

Many results have been obtained on the skew spectra and skew spectral radii of oriented graphs [2, 5, 4, 6, 7, 9].

Recently (in 2013) Cai et al. [3] defined a new type of skew laplacian matrix $\widetilde{SL}(D)$ of a digraph D as follows. Let $D^{+}(D)$ and $D^{-}(G)$ respectively be the diagonal matrices of vertex out degree and vertex in degree and let $A^{+}(D)=(a_{ij})$ and $A^{-}(D)=(a_{ij})$ respectively be the out-adjacency and in-adjacency matrix of a digraph D. If A(G) is the adjacency matrix of the underlying graph G of the digraph D, then it is clear that A(G) = $A^{+}(D)$ + $A^{-}(D)$ and S(D) = $A^{+}(D)$ - $A^{-}(D)$ where S(D) is the skew adjacency matrix of D. Therefore, following the definition of Laplacian matrix of a graph, called the matrix

$$\widetilde{SL}(D) = (D^{+}(D) - D^{-}(D)) - (A^{+}(D) - A^{-}(D))$$

$$= \widetilde{D}(D) - S(D)$$

where $\widetilde{D}(D) = D^{+}(D) - D^{-}(D)$ as the skew signless Laplacian matrix of the digraph D where as the matrix

$$SSL(D) = (D^{+}(D) - D^{-}(D)) + (A^{+}(D) - A^{-}(D))$$

$$= \widetilde{D}(D) + S(D)$$

It is clear that both the matrices $\widetilde{SL}(D)$ *and* $SSL(D)$ are not symmetric, so their eigen values need not be real. In this paper we will confine ourselves for the matrix $SSL(D)$ *and* we have the following observation for the matrix $SSL(D)$.

Theorem 1.1.

(i) $v_1, v_2, \ldots, v_n$ are the eigen values of $SSL(D)$, then $\sum_{i=1}^{n} v_i = 0$

(ii) 0 is an eigen value of $SSL(D)$ with multiplicity p, where p is the number of components of D with all ones vector (1, 1, 1,…, 1) as the corresponding eigen vector.

Following the definition of matrix energy given by Nikifrov and Cai et al. [3] defined the skew laplacian energy of a digraph D, as the sum of the absolute values of the eigen values of the matrix $\tilde{SL}(D)$ and obtained various bounds.

In this paper, we will confine ourselves to the definition of skew signless laplacian energy of a digraph and obtained various results in this direction.

Definition 1.2. Skew signless laplacian energy of a digraph. Let D be a oriented graph of order n with m arcs and having skew signless laplacian eigen values $(\mu_1, \mu_2, \ldots, \mu_n)$. The skew signless laplacian energy of D is denoted by $SSLE(D)$ and is defined as

$$SSLE(D) = \sum_{i=1}^{n} | \mu_i |$$

The idea of defining energy in such a way is to conceive a graph energy like quantity for a digraph, that instead of skew signless adjacency eigen values is defined in terms of skew signless laplacian eigen values and that hopefully would preserve the main features of the original graph energy. The definition of $SSLE(D)$ was therefore so chosen that all the properties possessed by graph energy should be preserved.

In [8], we show that every even positive integer is indeed the skew Laplacian energy of some digraph.

Theorem 1.3. Every even positive integer 2(n - 1) is the skew Laplacian energy of a directed star.

We prove the following main result.

In this theorem we prove that skew signless Laplacian energy of a complete bipartite graph is an even number.

Theorem 1.4. Every positive integer 4(n-1) is the skew signless Laplacian energy of a complete oriented bipartite graph $K_{n,2}$.

Proof. Let $V(K_{n,2}) = \{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}\}$ be the vertex set of $K_{n,2}$. If $V_1 = \{v_1, v_2, \ldots, v_n\}$ is the partite set and $V_2 = \{v_{n+1}, v_{n+2}\}$ is the another partite set of the complete directed bipartite graph orient all edges towards $V_1 = \{v_1, v_2, \ldots, v_n\}$ from $V_2 = \{v_{n+1}, v_{n+2}\}$. Then

$$S(K_{n,2}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & -1 \\ 0 & 0 & 0 & & -1 & -1 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \end{bmatrix}$$

$$D(K_{n,2}) = \begin{bmatrix} -2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -2 & & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 0 & 0 & 0 & \ldots & n & 0 \\ 0 & 0 & 0. & \ldots. & 0 & n \end{bmatrix}$$

Therefore, $SSL(D) = (D^{+}(D) - D^{-}(D)) + (A^{+}(D) - A^{-}(D))$

$= D(D) + S(D)$

Hence, $$SSL(D) = \begin{bmatrix} -2 & 0 & 0 & \cdots & -1 & -1 \\ 0 & -2 & 0 & \cdots & -1 & -1 \\ 0 & 0 & -2 & & -1 & -1 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 1 & 1 & 1 & \cdots & n & 0 \\ 1 & 1 & 1 & \cdots & 0 & n \end{bmatrix}$$

By direct calculation it can easily be seen that skew signless Laplacian characteristic polynomial of this matrix is $x[x-(n-2](x-2)^{n-1}(x-n)$. Therefore it is easy to see that the skew signless Laplacian eigen values of this matrix are ($n-2, 0, (-2)^{n-1}, n$) and so $SSL(E(K_{n,2})) = 4(n-1)$.

On the other hand, we orient all the edges from V_2 to V_1 then it can be seen that

$$SSL(K_{n,2}) = \begin{bmatrix} 2 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 2 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 2 & & 1 & 1 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ -1 & -1 & -1 & \cdots & -n & 0 \\ -1 & -1 & -1 & \cdots & 0 & -n \end{bmatrix}$$

having the skew signless Laplacian characteristic polynomial of this matrix is $x[x+(n-2](x+2)^{n-1}(x+n)$ and so eigen values (-($n-2$), 0, $(2)^{n-1}, -n)$, so $SSLE(K_{n,2}) = 4(n-1)$. Thus for a complete oriented bipartite graph $K_{n,2}$, we have $SSLE(K_{n,2}) = 4(n-1)$.

Example 1.5. Let D = $K_{3,2}$ be a digraph as shown below with partite sets $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and $V_2 = \{u_1, u_2\}$ oriented all edges from V_2 to V_1.

Clearly,

$$D(K_{5,2}) = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad \text{and}$$

$$S(K_{5,2}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 5 \end{bmatrix}$$

Then, $SSL(D) = (D^+(D) - D^-(D)) + (A^+(D) - A^-(D))$

$= D(D) + S(D)$

Therefore , $$SSL(K_{5,2}) = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & -2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 5 \end{bmatrix}$$

Hence skew Laplacian spectrum of $SSL(K_{5,2}) = (3, 0, -2, -2, -2, -2, 5)$ and $SSLE(K_{5,2}) = 4(n-1) = 4(5-1).$ = 16. Similarly the skew signless laplacian spectrum of $SSLE(K_{5,2}) = (-3, 0, 2, 2, 2, 2, -5)$ if all edges are oriented from V_1 to V_2. Hence $SSLE(K_{5,2}) = 4(n-1) = 4(5-1).$ =16= 3+0+2+2+2+2+5.

Corollary 1.6. If D is a directed star, then the skew signless Laplacian energy of D is 2(n-1).

Proof. Let $V(K_{1,n}) = \{u_1, u_2, \ldots, u_n, u_{n+1}\}$ be the vertex set of $K_{1,n}$. If u_{n+1} is the center of $K_{1,n}$ Orient all the edges towards u_{n+1}. Then

$$S(K_{1,n}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & & 0 & 1 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & -1 & -1 & \cdots & 0 & -n \end{bmatrix}$$

$$D(K_{1,n}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0. & \ldots. & 0 & -n \end{bmatrix}$$

Therefore, we have $SSL(D) = (D^{+}(D) - D^{-}(D)) + (A^{+}(D) - A^{-}(D))$

$= D(D) + S(D)$

Hence, $$SSL(D) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & & 0 & 1 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & -1 & -1 & \cdots & 0 & -n \end{bmatrix}$$

It is easy to see that the eigen values of this matrix are (-(n-1), 0, 1^{n-1}**)** and so SSLE**(**$K_{1,n}$) = 2(n-1). On the otherhand, if we orient the edges away from u_{n+1}, then it can easily be seen that

$$SSL(D) = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & -1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & -1 & & 0 & -1 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & & & & \\ 0 & 0 & 0 & \cdots & -1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & n \end{bmatrix}$$

having eigen values ((n-1), 0, $(-1)^{n-1}$**).** So SSLE**(**$K_{1,n}$) = 2(n-1)**.** Thus for directed star $K_{1,n}$, we have SSLE**(**$K_{1,n}$) = 2(n-1)**.**

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