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Some Extensions and Generalizations of the Bounds for the Zeros of a Polynomial with Restricted Coefficients

M.H.Gulzar [1] **, Ajaz Wani** [2]

[1,2] *Department of Mathematics, University of Kashmir, Srinagar (India)*

ABSTRACT

In this paper we give some extensions and generalizations of the bounds for the zeros of a polynomial with restricted coefficients.

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I.INTRODUCTION

Regarding the location of the zeros of a polynomial with restricted coefficients
we have the following famous result known as the Enestrom-Kakeya Theorem [7].

Theorem 1.1. All the zeros of a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ with

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0$$

lie in $|z| \leq 1$.

The above result has been extended and generalized in various ways by the researchers (see [1-4],[6-7], etc.).Recently M.Al-Hawari and F.M.Al-Askar [5] claim to have proved the following result:

Theorem 1.2. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and λ with $k_1 \geq 1, k_2 \geq 1, 0 < \rho \leq 1, 0 < \lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq ...... \geq k_2 a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 \geq 0 .$$

Then all the zeros of P(z) lie in the closed disk

$$|z + k_1 - 1| \leq k_1 + k_2 \frac{a_\lambda}{a_n} + \frac{2a_0}{a_n}(1-\rho).$$

They also derived the following corollaries from Theorem 1.2:

Corollary 1.1 . Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers ρ and λ with $0<\rho\le 1, 0<\lambda\le n-1$,

$$a_{n-1}\ge a_{n-2}\ge ......\ge a_n\ge a_{\lambda-1}\ge ......\ge a_1\ge \rho a_0\ge 0 \ .$$

Then all the zeros of P(z) lie in the closed disk

$$\left| z+\frac{a_{n-1}}{a_n}-1\right| \le \frac{a_{n-1}}{a_n}+1+2\frac{a_0}{a_n}(1-\rho) .$$

Corollary 1. 2 . Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_{n-1}\ge a_{n-2}\ge ......\ge a_\lambda\ge a_{\lambda-1}\ge ......\ge a_1\ge a_0\ge 0 \ .$$

Then all the zeros of P(z) lie in the closed disk

$$|z|\le 1+\frac{a_\lambda}{a_n} .$$

Corollary1. 3 : Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, ρ and λ with $k_1\ge 1, 0<\rho\le 1, 0<\lambda\le n-1$,

$$k_1 a_n\ge a_{n-1}\ge ......\ge a_\lambda\ge a_{\lambda-1}\ge ......\ge a_1\ge \rho a_0\ge 0 \ .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k_1-1|\le k_1+\frac{a_\lambda}{a_n}+2\frac{a_0}{a_n}(1-\rho) .$$

Corollary 1.4 . Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some integer λ with $0<\lambda\le n-1$,

$$a_n{}^2 \geq a_{n-1} \geq ...... \geq a_n a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq a_0 \geq 0 \; .$$

Then all the zeros of P(z) lie in the closed disk

$$|z + a_n - 1| \leq a_n + a_\lambda \, .$$

Unfortunately, in the proof of Theorem 1.2, the bound for the zeros has been wrongly calculated and consequently the bounds in the corollaries also are not correct. The correct forms of the above results are given below.

II.MAIN RESULTS

Theorem 2.1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0 < \rho \leq 1, 0 < \lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq ...... \geq k_2 a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 \geq 0 \; .$$

Then all the zeros of P(z) lie in the closed disk

$$|z + k_1 - 1| \leq k_1 + 2(k_2 - 1)\frac{a_\lambda}{a_n} + \frac{2a_0}{a_n}(1 - \rho).$$

Instead of proving Theorem 2.1, we prove the following more general result.

Theorem 2.2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0 < \rho \leq 1, 0 < \lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq ...... \geq k_2 a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 \; .$$

Then all the zeros of P(z) lie in the closed disk

$$|z + k_1 - 1| \leq \frac{1}{|a_n|}\{k_1 a_n + 2(k_2 - 1)|a_\lambda| - \rho(a_0 + |a_0|) + 2|a_0|\} \, .$$

Proof of Theorem 2.2 . Consider the polynomial

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$$F(z) = (1-z)P(z)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + ... + (a_1 - a_0)z + a_0$$
$$= -a_n z^{n+1} - k_1 a_n z^n + a_n z^n + (k_1 a_n - a_{n-1})z^n + ... + a_{\lambda+2} - k_2 a_{\lambda+1})z^{\lambda+2} + (a_{\lambda+1} - k_2 a_\lambda)z^{\lambda+1}$$
$$+ (k_2 - 1)a_\lambda z^{\lambda+1} + (k_2 a_\lambda - a_{\lambda-1})z^\lambda - (k_2 - 1)a_\lambda z^\lambda + (a_{\lambda-1} - a_{\lambda-2})z^{\lambda-1} + ... + (a_2 - a_1)z^2$$
$$+ (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0 .$$

Then for $|z| > 1$, we have , by using the hypothesis

$$|F(z)| = | \ -a_n z^{n+1} - k_1 a_n z^n + a_n z^n + (k_1 a_n - a_{n-1})z^n + ... + (a_{\lambda+2} - a_{\lambda+1})z^{\lambda+2} + (a_{\lambda+1} - k_2 a_\lambda)z^{\lambda+1}$$

$$+ (k_2 - 1)a_\lambda z^{\lambda+1} + (k_2 a_\lambda - a_{\lambda-1})z^\lambda - (k_2 - 1)a_\lambda z^\lambda + (a_{\lambda-1} - a_{\lambda-2})z^{\lambda-1}$$
$$+ .... + (a_2 - a_1)z^2 + (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0 \ |$$

$$\geq |a_n||z|^n \Bigg[|z + k_1 - 1| - \frac{1}{|a_n|} \Big\{ |k_1 a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| . \frac{1}{|z|} + ... + |a_{\lambda+2} - a_{\lambda+1}| . \frac{1}{|z|^{n-\lambda-2}}$$

$$+ |a_{\lambda+1} - k_2 a_\lambda| . \frac{1}{|z|^{n-\lambda-1}} + (k_2 - 1)|a_\lambda| . \frac{1}{|z|^{n-\lambda-1}} + |k_2 a_\lambda - a_{\lambda-1}| . \frac{1}{|z|^{n-\lambda}}$$
$$+ (k_2 - 1)|a_\lambda| . \frac{1}{|z|^{n-\lambda}} + |a_{\lambda-1} - a_{\lambda-2}| . \frac{1}{|z|^{n-\lambda+1}} + ... + |a_2 - a_1| . \frac{1}{|z|^{n-2}}$$
$$+ |a_1 - \rho a_0| . \frac{1}{|z|^{n-1}} + (1 - \rho)|a_0| . \frac{1}{|z|^{n-1}} + |a_0| . \frac{1}{|z|^{n-1}} \ \Bigg]$$

$$> |a_n||z|^n \Bigg[|z + k_1 - 1| - \frac{1}{|a_n|} \{ k_1 a_n - a_{n-1} + a_{n-1} - a_{n-2} + ... + a_{\lambda+2} - a_{\lambda+1}$$

$$+ a_{\lambda+1} - k_2 a_\lambda + (k_2 - 1)|a_\lambda| + k_2 a_\lambda - a_{\lambda-1} + (k_2 - 1)|a_\lambda|$$
$$+ a_{\lambda-1} - a_{\lambda-2} + ... + a_2 - a_1 + a_1 - \rho a_0 + (1 - \rho)|a_0| + |a_0| \ \Bigg]$$

$$=|a_n||z|^n\left[|z+k_1-1|-\frac{1}{|a_n|}\{k_1a_n+2(k_2-1)|a_\lambda|-\rho(a_0+|a_0|)+2|a_0|\right]$$

$$>0$$

if

$$|z+k_1-1|>\frac{1}{|a_n|}\{k_1a_n+2(k_2-1)|a_\lambda|-\rho(a_0+|a_0|)+2|a_0|\}.$$

This shows that those zeros of F(z)whose modulus is greater than 1 lie in

$$|z+k_1-1|\le\frac{1}{|a_n|}\{k_1a_n+2(k_2-1)|a_\lambda|-\rho(a_0+|a_0|)+2|a_0|\}.$$

But the zeros of F(z)whose modulus is less than or equal to 1 already satisfy the above inequality.

Hence it follows that all the zeros of F(z) and therefore P(z) lie in

$$|z+k_1-1|\le\frac{1}{|a_n|}\{k_1a_n+2(k_2-1)|a_\lambda|-\rho(a_0+|a_0|)+2|a_0|\}.$$

Remark 2.1. If $a_0\ge 0$ in Theorem 2.2, we get Theorem 2.1.

Taking $k_1=\frac{a_{n-1}}{a_n}\ge 1, k_2=\frac{a_n}{a_\lambda}\ge 1$ in Theorem 2.2, we get the following

Corollary 2.1 . Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some positive number ρ and some integer λ with $0<\rho\le 1, 0<\lambda\le n-1$,

$$a_{n-1}\ge a_{n-2}\ge......\ge a_n\ge a_{\lambda-1}\ge......\ge a_1\ge\rho a_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$\left|z+\frac{a_{n-1}}{a_n}-1\right|\le\frac{1}{|a_n|}\{a_{n-1}+2(\frac{a_n}{a_\lambda}-1)|a_\lambda|-\rho(a_0+|a_0|)+2|a_0|\}.$$

If $a_0 \geq 0$ in Cor. 2.1, we get the correct form of Cor.1.1 as follows:

Corollary 2.2 . Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive number ρ and some integer λ with $0<\rho\leq 1, 0<\lambda\leq n-1$,

$$a_{n-1} \geq a_{n-2} \geq ...... \geq a_n \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 \geq 0 \ .$$

Then all the zeros of P(z) lie in the closed disk

$$\left|z+\frac{a_{n-1}}{a_n}-1\right| \leq \frac{a_{n-1}}{a_n}+2(1-\frac{a_\lambda}{a_n})+\frac{2a_0}{a_n}(1-\rho) \ .$$

Taking $k_1=k, k_2=\rho=1$ in Theorem 2.2, we get the following result of Aziz and zargar [2] which in turn gives the result of Joyal, Labelle and Rahman [6] for k=1 and then the Enestrom-Kakeya Theorem for $a_0 \geq 0$.

Corollary 2.3 . Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive number k,

$$ka_n \geq a_{n-1} \geq a_{n-2} \geq ...... \geq a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq a_0 \ .$$

Then all the zeros of P(z) lie in the closed disk

$$|z| \leq \frac{1}{|a_n|}\{ka_n - a_0 + |a_0|\} .$$

Taking $k_2=1$ in Theorem 2.2, we get the following result .

Corollary2. 4. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k, ρ and some integer λ with $k \geq 1, ,0<\rho\leq 1, 0<\lambda\leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq ...... \geq a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 \ .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k_1-1| \le \frac{1}{|a_n|}\{k_1 a_n - \rho(a_0+|a_0|)+2|a_0|\}.$$

For $a_0 \ge 0$ in corollary 2.4, we get the correct form of Cor.1.3 as follows.

Corollary2. 5. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1,ρ and some integer λ with with $k_1 \ge 1,,0<\rho\le 1,0<\lambda \le n-1,$

$$k_1 a_n \ge a_{n-1} \ge ...... \ge a_\lambda \ge a_{\lambda-1} \ge ...... \ge a_1 \ge \rho a_0 \ge 0 .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k_1-1| \le k_1 + 2\frac{a_0}{a_n}(1-\rho).$$

Taking $k_1 = k_2 = a_n$ and $\rho = 1$ in Theorem 2.2, we get the following result.

Corollary 2. 6. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some integer λ with $0<\lambda \le n-1,$

$$a_n{}^2 \ge a_{n-1} \ge ...... \ge a_n a_\lambda \ge a_{\lambda-1} \ge ...... \ge a_1 \ge a_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+a_n-1| \le \frac{1}{|a_n|}\{a_n + 2(a_n-1)|a_\lambda| - a_0 + |a_0|\}.$$

For $a_0 \ge 0$ in corollary 2.6, we get the correct form of Cor.1.4 as follows.

Corollary 2. 7. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some integer λ with $0<\lambda \le n-1,$

$$a_n{}^2 \ge a_{n-1} \ge ...... \ge a_n a_\lambda \ge a_{\lambda-1} \ge ...... \ge a_1 \ge a_0 \ge 0 .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+a_n-1| \le 1+2a_\lambda - \frac{2a_\lambda}{a_n}.$$

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