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A New Generalized Fuzzy Information Measure and Its Properties

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ABSTRACT

In this paper we propose a new two parametric generalized measure of fuzzy entropy of order α and type β and study its properties. Also, we study the behaviour of the new proposed measure with the numerical illustration.

Keywords- Shannon's entropy, Fuzziness, Fuzzy measure of information, Fuzzy set and Membership function.

I.INTRODUCTION

Fuzziness is a feature of imperfect information that results from the lack of crisp distinction between the elements belonging and not belonging to a set that is the boundaries of the set under consideration are not sharply defined. A measure of fuzziness often used and cited in the literature is the entropy that was first conceived by Lofti A. Zadeh [1], Professor at the University of Berkley, in 1965. The name entropy was chosen due to an intrinsic similarity of equations to the ones in the Shannon entropy. However, the two functions measure fundamentally different types of uncertainty. Basically, the Shannon [2] entropy measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment.

The concept of fuzziness has been applied to apparently all the phenomena: engineering, medicine, computer science and decision making, fuzzy traffic control, fuzzy aircraft control, etc. Different information measures were considered and investigated by several authors. Some of them are: Aczel. J. [3], Kapur J. N. [4], Khan A. B., Autur R. and Ahmad H. [5], Van Der Lubbe J. C. A. [6], Renyi. [7], Ashiq H. B. and M. A. K. Baig [8, 9] etc. There are situations where probabilistic measures of entropy do not work. To deal with such situations, instead of taking the probability, the idea of fuzziness can be explored.

1.1 Preliminaries

In this section, we introduce some well-known concepts and the notations related to fuzzy entropy measure. We will also focus on the theory of fuzzy sets.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universal set defined in the universe of discourse and it includes all possible elements related to the given problem then a fuzzy subset of universe X is defined as:

$$A = \{(x_i, \mu_A(x_i)) : x_i \in X,\ \mu_A(x_i) \in [0,1]\}$$

where $\mu_A(x_i)$ is a membership function (characteristic function or discrimination function) and gives the degree of belongingness of the element $'x_i'$ to the set 'A'. If every element of the set 'A' is '0' or '1', there is no uncertainty about it and the set is said to be a crisp set. On the other hand, a fuzzy set 'A' is defined by a characteristic function $\mu_A(x_i) = \{x_1, x_2, ..., x_n\} \to [0,1]$.We shall require some concepts related to fuzzy sets that are given below.

Suppose *A* and *B* are two fuzzy subsets of *X* with membership functions $\mu_A(x)$ and $\mu_B(x)$ where *x* is an arbitrary element of *X*.

- Sum of A and B, (A+B) is defined as:

  $$\mu_{A+B}(x_i) = \mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i)\mu_B(x_i)\ ;\ \forall x_i \in X$$

- Product of A and B, (AB) is defined as:

  $$\mu_{AB}(x_i) = \mu_A(x_i)\mu_B(x_i)\ ;\ \forall x_i \in X$$

- Equality of A and B, (A=B) is defined as:

  $$\mu_A(x_i) = \mu_B(x_i)\ ;\ \forall x_i \in X$$

- Containment of A and B, (A$\subset$B) is defined as:

  $$\mu_A(x_i) \le \mu_B(x_i)\ ;\ \forall x_i \in X$$

- Complement of A, A' is defined as:

  $$\mu_{A'}(x_i) = 1 - \mu_A(x_i)\ ;\ \forall x_i \in X$$

- Union of A and B, (A$\cup$B) is defined as:

  $$\mu_{A\cup B}(x_i) = Max\{\mu_A(x_i), \mu_B(x_i)\}\ ;\ \forall x_i \in X$$

- Intersection of A and B, (A$\cap$B) is defined as:

  $$\mu_{A\cap B}(x_i) = Min\{\mu_A(x_i), \mu_B(x_i)\}\ ;\ \forall x_i \in X$$

1.1.1 Shannon's Entropy

Let $X = (x_1, x_2, ..., x_n)$ be a discrete random variable with probability distribution $P = (p_1, p_2, ..., p_n)$ such that $p_i \ge 0\ ;\ \forall i = 1, 2, ..., n$ and $\sum_{i=1}^{n} p_i = 1$. The information measure given by Shannon [2] is called entropy and is defined as follows:

$$H(P) = -\sum_{i=1}^{n} p_i \log_D p_i \tag{1}$$

De-Luca and Termini [10] introduced a measure of fuzzy entropy corresponding to Shannon's [2] measure of entropy, given as:

$$H(A) = -\sum_{i=1}^{n} [\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))]$$

Bhandari and Pal [11] gave some new measures corresponding to Renyi's [7] entropy measure of order α after surveying the literature available on information measures of fuzzy sets. They suggested the amount of ambiguity of order α should be:

$$H_\alpha(A)=\frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\right];\ \alpha\neq 1,\alpha\geq 0$$

Corresponding to Havrda and Charvat's [12] measure of information, Kapur [13] suggested the following fuzzy measure:

$$H_\alpha(A)=\frac{1}{1-\alpha}\sum_{i=1}^{n}\left[\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha-1\right];\ \alpha\neq 1,\alpha\geq 0$$

Corresponding to Sharma and Taneja's [14] information measure of degree (α,β), Kapur [13] proposed the following fuzzy measure:

$$H_{\alpha,\beta}(A)=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\left[\left\{\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\right\}-\left\{\mu_A^\beta(x_i)+(1-\mu_A(x_i))^\beta\right\}\right];$$

$$\alpha\geq 1,\beta\leq 1\ or\ \beta\geq 1,\alpha\leq 1$$

Corresponding to Campbell's [15] entropy measure, Prakash and Sharma [16] has taken the following fuzzy measure:

$$H_\alpha(A)=\frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n}\left\{\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\right\}^\alpha\right];\ \alpha\neq 1,\alpha\geq 0$$

Ashiq and Baig's [8, 9] defined the following two generalized fuzzy measures as:

$$H_\alpha^\beta(A)=\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right\}\right];\ 0<\alpha<1,0<\beta\leq 1$$

$$H_\alpha^\beta(A)=\frac{\beta}{\beta-\alpha}\left[\sum_{i=1}^{n}\left\{\mu_A^{\alpha\beta}(x_i)+(1-\mu_A(x_i))^{\alpha\beta}\right\}\right];\ 0<\alpha<1,0<\beta\leq 1,\beta>\alpha$$

In the next section, we shall define a new generalized fuzzy information measure of order α and type β and also show that it is a valid measure of fuzzy entropy.

II.NEW GENERALIZED FUZZY INFORMATION MEASURE AND ITS PROPERTIES

Define a two parametric new generalized fuzzy information measure as:

$$H_\alpha^\beta(A)=\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}\right];\ 0<\alpha<1,0<\beta\leq 1;\beta>\alpha \qquad (2)$$

In order to prove that (2) is a valid fuzzy measure we show that the following four properties that is, sharpness, maximality, resolution and symmetry are satisfied.

2.1Sharpness: $H_\alpha^\beta(A)$ is minimum if and only if A is a crisp set

i.e., $H_\alpha^\beta(A)=0$ iff $\mu_A(x_i)=0\ or\ 1;\ \forall\ i=1,2,\ldots,n$

Proof: Let us suppose $H_\alpha^\beta(A)=0$, i.e.,

$$\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}\right]=0$$

$$\Rightarrow \log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}\right]=0 \quad (3)$$

We know that $\log_D x=0$ if $x=1$. Using this result in (3), we get $\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}\right]=\log_D 1$

$$\Rightarrow\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}\right]=1$$

$$\Rightarrow\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}=n \quad (4)$$

Since, $0<\alpha<1, 0<\beta\le 1.$, (4) will hold when either $\mu_A(x_i)=1$ or $\mu_A(x_i)=0$; $\forall\, i=1,2,\ldots,n.$

Conversely suppose, $\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}\right]=1,$; $\forall\, 0<\alpha<1, 0<\beta\le 1; \beta>\alpha.$

$$\Rightarrow \log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}\right]=0$$

$$\Rightarrow\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}\right]=0$$

$$\Rightarrow H_\alpha^\beta(A)=0$$

Hence, $H_\alpha^\beta(A)=0$ if and only if A is a crisp set.

2.2 Maximality: $H_\alpha^\beta(A)$ is maximum if and only if A is most fuzzy set.

Proof: We have

$$H_\alpha^\beta(A)=\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}\right];\ \forall\, 0<\alpha<1, 0<\beta\le 1; \beta>\alpha \quad (5)$$

Now differentiating equation (5) with respect to $\mu_A(x_i)$ we get

$$\frac{\partial H_\alpha^\beta(A)}{\partial\mu_A(x_i)}=\beta^2\left[\frac{\mu_A^{\beta(1-\alpha)-1}(x_i)-(1-\mu_A(x_i))^{\beta(1-\alpha)-1}}{\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}}\right]$$

Let $0\le\mu_A(x_i)<0.5$ then

$$\frac{\partial H_\alpha^\beta(A)}{\partial \mu_A(x_i)} > 0\ ;\ \forall\, 0<\alpha<1, 0<\beta\le 1; \beta>\alpha$$

Hence, $H_\alpha^\beta(A)$ is an increasing function of $\mu_A(x_i)$ whenever, $0 \le \mu_A(x_i) < 0.5$.

Similarly, for $0.5 < \mu_A(x_i) \le 1$, we have

$$\frac{\partial H_\alpha^\beta(A)}{\partial \mu_A(x_i)} < 0\ ;\ \forall\, 0<\alpha<1, 0<\beta\le 1; \beta>\alpha$$

Hence, $H_\alpha^\beta(A)$ is a decreasing function of $\mu_A(x_i)$ whenever, $0.5 < \mu_A(x_i) \le 1$

And for $\mu_A(x_i) = 0.5$

$$\frac{\partial H_\alpha^\beta(A)}{\partial \mu_A(x_i)} = 0\ ;\ \forall\, 0<\alpha<1, 0<\beta\le 1; \beta>\alpha$$

Thus, $H_\alpha^\beta(A)$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5$. This implies $H_\alpha^\beta(A)$ is maximum iff A is most fuzzy set that is, $\mu_A(x_i) = 0.5\ ;\ \forall\, i = 1, 2, \ldots, n.$

2.3 Resolution: $H_\alpha^\beta(A) \ge H_\alpha^\beta(A^*)$, where A^* is sharped version of A.

Proof: Since $H_\alpha^\beta(A)$ is an increasing function of $\mu_A(x_i)$ whenever, $0 \le \mu_A(x_i) < 0.5$ and is a decreasing function of $\mu_A(x_i)$ whenever, $0.5 < \mu_A(x_i) \le 1$, therefore

$$\mu_{A^*}(x_i) \le \mu_A(x_i)$$

$$\Rightarrow H_\alpha^\beta(A^*) \le H_\alpha^\beta(A) \text{ in } [0, 0.5) \tag{6}$$

Also,

$$\mu_{A^*}(x_i) \ge \mu_A(x_i)$$

$$\Rightarrow H_\alpha^\beta(A^*) \le H_\alpha^\beta(A) \text{ in } (0.5, 1] \tag{7}$$

Taking equation (6) and (7) together, we get

$$\Rightarrow H_\alpha^\beta(A^*) \le H_\alpha^\beta(A)$$

2.4 Symmetry: $H_\alpha^\beta(A) = H_\alpha^\beta(A')$, where A' is the complement of A.

Proof: It may be noted that from the definition of $H_\alpha^\beta(A)$ and $\mu_{A'}(x_i) = 1 - \mu_A(x_i) \forall\, x_i \in X$, we conclude that $H_\alpha^\beta(A) = H_\alpha^\beta(A')$.

Since, $H_\alpha^\beta(A)$ satisfies all the properties of fuzzy entropy, thus it is a valid measure of fuzzy entropy.

III.ILLUSTRATION

In this section, we demonstrate numerically that the proposed fuzzy measure is valid. To do so, we verify the properties (that is, sharpness, maximality, resolution and symmetry) at different values of α , β and membership function $\mu_A(x_i)$.

3.1 Sharpness

Table 3.1: Behaviour of $H_\alpha^\beta(A)$, when $\mu_A(x_i)=1$ and $\mu_A(x_i)=0$ with respect to α and β.

α	β	$\mu_A(x_i)$	$H_\alpha^\beta(A)$	$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.1	0.2	1	0	0	0
0.3	0.4	1	0	0	0
0.5	0.6	1	0	0	0
0.7	0.9	1	0	0	0
0.9	1.0	1	0	0	0

From the above table we conclude that $H_\alpha^\beta(A)$ is minimum (i.e., $H_\alpha^\beta(A)=0$) iff A is a crisp set (i.e., when $\mu_A(x_i)=0$ or $\mu_A(x_i)=1$).

3.2 Maximality

Table 3.2.1: At $0\le\mu_A(x_i)<0.5$ and with respect to α and β.

α	β	$\mu_A(x_i)$	$\frac{\partial H_\alpha^\beta(A)}{\partial\mu_A(x_i)}$
0.1	0.2	0	Inf
		0.15	0.0232
		0.30	0.00868
		0.45	0.00188

From the above table we conclude that $H_\alpha^\beta(A)$ is an increasing function (i.e., $\frac{\partial H_\alpha^\beta(A)}{\partial\mu_A(x_i)}>0$) of $\mu_A(x_i)$ whenever, $0\le\mu_A(x_i)<0.5$.

Table 3.2.2: $0.5<\mu_A(x_i)\le1$ and with respect to α and β.

x_i	α	β	$\mu_A(x_i)$	$\frac{\partial H_\alpha^\beta(A)}{\partial\mu_A(x_i)}$
1	0.1	0.2	0.65	-0.0061
2			0.73	-0.0105
3			0.80	-0.0164
4			1.0	-inf

From the above table we conclude that $H_\alpha^\beta(A)$ is a decreasing function (i.e., $\frac{\partial H_\alpha^\beta(A)}{\partial\mu_A(x_i)}<0$) of $\mu_A(x_i)$ whenever $0.5<\mu_A(x_i)\le1$.

For $\mu_A(x_i)=0.5$, $\alpha=0.1$ and $\beta=0.2$ we get $\frac{\partial H_\alpha^\beta(A)}{\partial\mu_A(x_i)}=0.$ (8)

Thus, from table (3.2.1), (3.2.2) and equation (8) we conclude that $H_\alpha^\beta(A)$ is a concave function with global maximum at $\mu_A(x_i)=0.5$.

3.3 Resolution

Table 3.3.1: At $[0,0.5)$ **and with** $\mu_A(x_i) \geq \mu_{A^*}(x_i)$ **.**

α	β	$\mu_A(x_i)$	$H_\alpha^\beta(A)$	$\mu_{A^*}(x_i)$	$H_\alpha^\beta(A^*)$
0.2	0.6	0.12	0.2431	0	0.1918
		0.23		0.15	
		0.36		0.31	
		0.49		0.44	

From above table we conclude that $H_\alpha^\beta(A^*) \leq H_\alpha^\beta(A)$ whenever $\mu_A(x_i) \geq \mu_{A^*}(x_i)$ in $[0,0.5)$.

Table 3.3.2: At $(0.5,1]$ **and with** $\mu_A(x_i) \leq \mu_{A^*}(x_i)$ **.**

α	β	$\mu_A(x_i)$	$H_\alpha^\beta(A)$	$\mu_{A^*}(x_i)$	$H_\alpha^\beta(A^*)$
0.2	0.6	0.65	0.2033	0.70	0.1636
		0.83		0.83	
		0.89		0.94	
		0.96		1.0	

From above table we conclude that $H_\alpha^\beta(A^*) \leq H_\alpha^\beta(A)$ whenever $\mu_A(x_i) \leq \mu_{A^*}(x_i)$ in $(0.5,1]$

Thus, from table (3.3.1) and (3.3.2) we conclude that $H_\alpha^\beta(A^*) \leq H_\alpha^\beta(A)$ where A^* is sharped version of A.

3.4 Symmetry

Table 4:

α	β	$\mu_A(x_i)$	$H_\alpha^\beta(A)$	$1-\mu_A(x_i)$	$H_\alpha^\beta(A')$
0.2	0.6	0.65	0.2074	0.35	0.2074
		0.78		0.22	
		0.89		0.11	
		0.96		0.04	

From the above table we conclude that $H_\alpha^\beta(A)=H_\alpha^\beta(A')$, where A' is the complement of A.

IV.CONCLUSION

In this paper, we introduced a new generalized entropy measure i.e., $H_\alpha^\beta(A)$ of order α and type β with the proof of its validity. Also, we studied the monotonic behaviour of $H_\alpha^\beta(A)$ with respect to parameters α and β. Some of the interesting properties of $H_\alpha^\beta(A)$ have also been studied with the help of hypothetical data.

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