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Convergence of the multi-step binomial model in the binomial market model to the Black-Scholes Financial model

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ABSTRACT

The basic concepts in probability and stochastic calculus, We study convergence of the multi-step binomial model binomial market model to the Black-Scholes Financial model. Precisely, by using the De Moivre Laplace Central Limit Theorem, We show that Cox-Ross-Rubinstein's formula for the price of a European call option in the multi-step binomial model converges in distribution to the celebrated Black-Scholes formula for a European call option price in Black-Scholes Financial model.

Keywords: Cox-Ross-Rubinstein's formula, Multi-step Binomial model, Converges in distribution, Black-Scholes formula

I INTRODUCTION

Financial Mathematics most of the time deals with the issue of pricing financial assets such as financial derivatives. A central concept is that of arbitrage, i.e., without investing money in the market, the arbitrageur makes a risk-free profit. Pricing in a no-arbitrage setting can be set as a mathematical problem. This allows for the computation of explicit prices for financial assets in some specific cases. In this paper, we consider such a case, namely the pricing of options.

Option pricing theory got attention after publication of the land mark paper titled with "The Pricing of Options and corporate liability " which was published in the Journal of Political Economy by Fisher Black and Myron Sholes in 1973 [11]. In this paper we discuss two common option pricing model: the Black-Scholes and the binomial option pricing model, and the convergence of the binomial model to the Black-Scholes model.

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II PRELIMINARY CONCEPT

In this Section we will review useful concepts of Probability theory and tools of basic Stochastic Calculus for this peper. Definitions, theorems, propositions, etc are mainly taken from references [2, 6, 8, 9, 12, 13].

2.1 Review of probability theory

Definition.2.1.1. Let $X: \Omega \to R$ be any discrete random variable with range $X(\Omega) = \{ x_1, x_2, x_3, \dots, x_n \}$ or $X(\Omega) = \{ x_1, x_2, x_3, \dots, x_n, \dots \}$. Then the expectation of X is defined as

$$\mathbb{E}(X) = \sum_k x_k P(x_k) \quad \text{when this sum is finite}. \qquad (2.1.1)$$

If $f : \mathbb{R} \to \mathbb{R}$ is any function, then

$$\mathbb{E}(f(X)) = \sum_k F(x_k) P(x_k) \qquad (2.1.2)$$

whenever this sum exists.

Next, the variance of X is defined as

$$\mathbb{V}ar(X) := \sum_k \mathbb{E}(X - \mathbb{E}X)^2 P(x_k) \qquad (2.1.3)$$

If this sum is finite too.

Preposition.2.1.2.(The Binomial Distribution). A discrete random variable X has a binomial distribution with parameters n and p, with $n \in N$ and $0 < p < 1$, and we denote $X \sim B\,(n, p)$, if X is the number of successes obtained after n independent identically repeated Bernoulli trials, each with the same parameter p. Then,

1. The discrete random variable with range $X(\Omega) = \{ 0, 1, 2, 3, \dots, n \}$
2. The Probability mass function of discrete random variable X is given by
   $\mathbb{P}[X = k] = \binom{n}{k} P^k (1-p)^{n-k}$,$K = 0,1,2, \dots$
3. The expectation discrete random variable X is given by $\mathbb{E}(X) = np$ and the Variance discrete random variable X is given by $\mathbb{V}ar(X) = np(1 - p)$.
4. And for any function $f : \mathbb{R} \to \mathbb{R}, \mathbb{E}\big(f(X)\big) = \binom{n}{k} f(k) P^k (1-p)^{n-k}$.

Definition.2.1.3. Let $X : \Omega \to \mathbb{R}$ be an absolutely continuous random variable with Probability mass function f_X .If the improper integral $\int_{\mathbb{R}}^{\cdot} |x| f_X(x) dx$ is finite, then the expectation of continuous random variable X is defined as

$$\mathbb{E}(X) = \int_{\mathbb{R}}^{\cdot} x f_X(x) dx$$

Let $g : \mathbb{R} \to \mathbb{R}R \to R$ any function, then we define the expectation of $g(X)$ as

$$\mathbb{E}(g(X)) = \int_{\mathbb{R}}^{\cdot} g(x) f_X(x) dx \quad whenever\ this\ integral\ exists.$$

The variance of X is also defined as

$$\mathbb{V}ar(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \int_{\mathbb{R}}^{\cdot} (X - \mathbb{E}X)^2 f_X(x) dx. \quad If\ this\ integral\ exist$$

Definition.2.1.4. (The Gaussian or normal distribution). Let $\mu, \sigma \in \mathbb{R}$ such that $\sigma > 0$. We say that a random variable X on a probability space (Ω, F, P) has a "Gaussian or normal distribution" with parameters μ and σ, denoted $X \sim N(\mu, \sigma^2)$, if X is absolutely continuous with range $\mathbb{R}$ and with Probability mass function $f_{\mu,\sigma}$ given by

$$f_{\mu,\sigma}(t) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) \quad , \qquad t \in \mathbb{R}$$

When $\mu = 0$ and $\sigma = 1$, we say that $N(0,1)$ is called the "standard Gaussian or normal" random variable. It Probability mass function $f_{0,1}$ is usually denoted ϕ; i.e.

$$\phi(\mathrm{t}) = f_{0,1}(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad , \qquad t \in \mathbb{R}$$

And its Cumulative density distribution is usually denoted Φ; i.e.

$$\Phi(x) = \int_{-\infty}^{x} \phi(\mathrm{t})\mathrm{dt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\ \mathrm{dt}, \qquad x \in \mathbb{R}$$

Proposition.2.1.5.

1. For all $x \in \mathbb{R}_+, \Phi(-x) = 1 - \Phi(x)$
2. If $X \sim N(\mu, \sigma)$, then $\mathbb{E}(X) = \mu$ and $\mathbb{V}ar(X) = \sigma^2$
3. If $X \sim N(\mu, \sigma)$, then the scaled random variable $Z = \frac{X-\mu}{\sigma}$ is $N(0,1)$.
4. Let $X \sim N(\mu, \sigma)$, then $\mathbb{E}[e^x f(x)] = e^{\mu + \frac{\sigma^2}{2}}\ \mathbb{E}[f(x + \sigma^2)]$ for any non negative function f.

Definition.2.1.6.((Modes of convergence of random variables). Let X be any random variable and $(X_n)\ n \geq 1$ a sequence of random variables on the same probability space $(\Omega, F, \mathbb{P})$ with cumulative density function F_X and sequence of cumulative density function F_{Xn} respectively.

1. We say that $(X_n)_n$ converges almost surely (or strongly) to X, denoted $X_n \ \underline{a.s}\ $ X,

   If $\mathbb{P}\left(\left\{w\colon \lim_{n\to\infty} X_n(w) = X(w)\right\}\right) = 0$ which is equivalent to $\mathbb{P}\left(\left\{w\colon \lim_{n\to\infty} X_n(w) \neq X(w)\right\}\right) = 1$

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2. We say that $(X_n)_n$ converges in probability to X, denoted $X_n \xrightarrow{\mathbb{P}} X$, if for all $\epsilon > 0$, we have, $\lim_{n\to\infty} \mathbb{P}[|X_n - X > \epsilon = 0$.
3. And we say that $(X_n)_n$ converges in distribution (or weakly) to X, denoted , $X_n \xrightarrow{d} X$ or $X_n \Rightarrow X$, if , $\lim_{n\to\infty} F_{X_n}(x) = F_x(x), \ \forall x \in \mathbb{R}$.

Proposition2.1.7. Almost sure convergence implies convergence in probability, which in turn, implies convergence in distribution.

Definition2.1.8. (i.i.d. sequence of random variables). We say that a sequence of random variables $(Xn)_n$ is an i.i.d. sequence if for every pair of indices $i \neq j$, $X_i \sim X_j$ (i.e., X_i and X_j have the same distribution) and X_i Xi and X_j are independent.

Theorem2.1.9.(The Central Limit Theorem - CLT). Let $(Xn)_n \geq 1$ be an i.i.d. sequence of random variables with finite common expectation μ and finite common variance σ^2. Set $S_n := X_1 + X_2 + \cdots + X_n, for\ all\ n \geq 1$. Then we have

$$\frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} = X_n \xrightarrow{d} N(0,1) \quad i.e \lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq a\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{1}{2}x^2} \, dx$$

for all real number a.

$Proof$. See the proof of Theorem 11.12 in [8, pp. 500-501] for details.

Theorem2.1.10. Let X be a binomial random variable with parameters n and p, then

$$\frac{X - np}{\sqrt{np(1-p)}} = \xrightarrow{d} N(0,1)$$

2.2 Review of basic stochastic calculus

Definition 2.2.1. (Itó process). A real-valued stochastic process X_t is called an "Itó process" if there are two processes F_t in $\mathbb{L}^1[0,T]$ and G_t in $\mathbb{L}^2[0,T]$ such that for all times $0 \leq s \leq t \leq T$; we have

$$X_t = X_s + \int_s^t F_u du + \int_s^t G_u dW_u \tag{2.2.1}$$

In particular $s = 0$

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$$X_t = X_0 + \int_0^t F_u du + \int_0^t G_u dW_u \tag{2.2.2}$$

In this case we say that X_t has the "Stochastic differential "$dX_t = F_t dt + G_t dW_t$.

Proposition.2.2.2 (Martingale condition for Itó processes). Let X_t be Itó process as above. Then X_t is a $\mathbb{P}$-martingale with respect to natural filtration $\mathbb{F}^W$ if and only if $F_t = 0$ for all t almost sure.

Theorem.2.2.3 (Itó formula for functions of Itó processes). Let X_t be Itó process as above, and let $f: [0, T]\, x\, \mathbb{R} \to \mathbb{R}$ be a $C^{1,2}$ function such that $f(t, X_t) \in \mathbb{L}^2[0, T]$. Then the process $f(t, X_t)$ is an Itó process with stochastic differential,

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} {G_t}^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \frac{\partial f}{\partial x} dX_t \tag{2.2.3}$$

Definition.2.2.3. (Stochastic differential equation). A basic one-dimensional stochastic differential equation (abbreviated SDE) driven by a one-dimensional standard Brownian motion W_t is defined as a stochastic differential with an initial conditional in the form,

$$\begin{cases} dX_t = \mu(t, X_t) + \sigma(t, X_t) dW_t \\ X_0 = x_0 \in \mathbb{R} \end{cases} \tag{2.2.4}$$

The processes $\mu(t, X_t)$ and $\sigma(t, X_t)$ are called drift and discussion coefficients of the SDE.

An $\mathbb{F}_W$ adapted process X_t is a solution to this SDE if $X_0 = x_0$ and if there are two coefficients processes $\mu(t, X_t) \in \mathbb{L}^1[0, T]$ and $\sigma(t, X_t) \in \mathbb{L}^2[0, T]$. such that

$$X_t = X_0 + \int_0^t \mu(u, X_u) du + \int_0^t \sigma(u, X_u) dW_u \qquad \forall t \in \mathbb{R} \tag{2.2.5}$$

Existence and uniqueness for solutions of stochastic differential equations under certain conditions are discussed in [12], where explicit solutions are given for instance for linear SDE for which the coefficients are in the form $\mu(t, X_t) = \alpha_t + \beta_t X_t$ and $\sigma(t, X_t) = \lambda_t + \gamma_t X_t$ for some non-random functions of time α, β, λ and γ.

Definition.2.2.4. (Generator of an SDE). Consider any stochastic differential as in (2.2.4) above. The "generator" of this SDE is the operator denoted L, defined on the set of functions $f(t, x)$ in $C^{1,2}([0, T] \to \mathbb{R})$ by

$$L_t f(t, x) = \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 f}{\partial x^2} + \mu(t, x) \frac{\partial f}{\partial x} \qquad \forall (t, x) \in [0, T] x\mathbb{R} \tag{2.2.6}$$

Next, consider any real-valued bounded function r and h on $[0, T] \to \mathbb{R}$ and $\mathbb{R}$ respectively. For all

$(t, x) \in [0, T]x\mathbb{R}$, defined the function C by

$$C(t,x) = \mathbb{E}\left(e^{-\int_t^T r(u,X_u)du} h(X_T)|X_t = x\right) \tag{2.2.7}$$

For any Itó process X_t on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to the natural filtration $\mathbb{F}^W$ of the Brownian motion W_t.

Theorem.2.2.5.(Feynman-Kac's formula). X_t is a solution to the SDE (2.2.4) if and only if $C(t, x)$ solves the partial differential equation,

$$\begin{cases} \dfrac{\partial f(t,x)}{\partial t} + L_t\ f(t,x) = r(t,x)\, f(t,x) \\ f(x,T) = h(x) \end{cases} \tag{2.2.8}$$

III OPTIONS PRICING IN THE MULTI-STEP BINOMIAL MODEL

3.1 Probabilistic set-up of the model

In discrete-time setting, the multi-step binomial model is built as an iterated sequence of one-step binomial models as follows: consider a financial market with N trading dates in the future (typically N years,$N \geq 2$), starting from today at time $t = 0$. In this market model we assume two assets in trading:

- A ***risk-less*** asset such as a risk-free bond or bank account with price or balance $B_0 = 1$ Dollar (for simplicity) at time $t = 0$, which attracts annual compounding interest at constant rate $r > 0$. Hence its price (or balance) at any future time $t \in \mathbb{N}, 1 \leq t \leq N$ is $B_t = (1 + r)^t$.

- And a ***risky*** asset such as stock whose price at time $t = 0$, denoted S_0 , is a positive constant, known by all investors. But its future prices, denoted $S_t,\ 1 \leq t \leq N$, are random and satisfy the recursive dynamics

$$S_t = \begin{cases} uS_{t-1} = (1+U)S_{t-1} & \text{with probability } p_u \\ dS_{t-1} = (1+D)S_{t-1} & \text{with probability } p_d \end{cases} \tag{3.1}$$

Defined recursively at each time-step (from $t - 1\ to\ t$), under a given investors' feeling probability measure $\mathbb{P} = (p_u; p_d)$ assumed the same on every time-step possible movements of the stock price: up by factor $u = 1 + U$ or down by factor $d = 1 + D$, where $U := \frac{S_t - S_{t-1}}{S_{t-1}}$ and $D := \frac{S_t - S_{t-1}}{S_{t-1}}$ are constant upward and downward returns on the stock price at every time-step$[t - 1\ , t], 1 \leq t \leq N$. We assume $0 < d < 1 < u$. The illustration of this market model for three N = 3 time steps is given by the following figure.

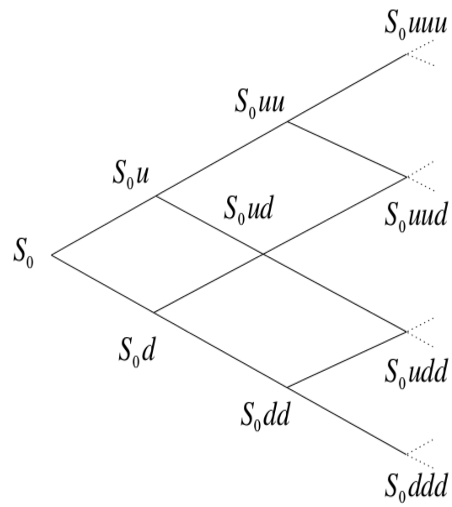

Figure 3.1 Illustration of multi-step binomial market model.

The modeling sample space is $\Omega = \{\omega_{1,}\omega_{2,}, \ldots, \omega_N\}$ where ω_t correspond to either the up scenario or the down scenario of the stock price at the end of each time-step $[t-1\,,t], 1\ \leq t\ \leq N$. Consider $\mathcal{F}\ := \mathrm{P}(\ \Omega)$, the power set of Ω , as the $\sigma - algebra$ on Ω . And we introduce here the natural filtration $\mathbb{F}\ := \mathcal{F}(t)_t$ of the stock prices process S_t , where $\mathcal{F}_{\ t}\ =\ (S_0,\ S_1, \ldots, S_t)$ represents the information on the stock prices up to and including time $t, 1\ \leq t\ \leq N$.

Remark 3.1: From the recursive dynamics of equation (3.1) above, it follows that if by any time step $t\ \leq N$ the stock prices have gone up k times and (hence) have gone down $t-k$ times, and then $S_t\ = u^k\ d^{t-k}S_0$ in particular we have $S_N\ = u^k\ d^{N-k}S_0$

Hence, let X_N be the random number of such up movements of the stock by step time N, then X_N is follows a binomial distribution with parameters N and p_u. Since S_N is therefore a function of X_t as $S_N\ = u^{X_N}\ d^{N-X_N}S_0$, it follows by Preposition.2.1.2.that

$$\mathbb{E}(S_N) = \sum_{k=0}^{N} \binom{N}{k} {p_u}^K {p_d}^{N-k} u^k\ d^{N-k} S_0 \tag{3.2}$$

Where $\mathbb{E}$ is the expectation under the probability measure$\mathbb{P}$.

3.2 Investment strategies and arbitrage

Definition ***3.2*** **(Portfolio)** A portfolio (investment strategy or trading opportunity) in this multi-step binomial model is any vector process $\varphi_t := (x_{\mathrm{t}},\ \mathrm{y}_{\mathrm{t}}) \in \mathbb{R}^2$ where x_{t} is the number of units of bond (or the bank account) and y_{t} is the number of units (or shares) of stock that an investor holds both from time $t-1$ to time t.

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Definitation ***3.3*** **(Investor's wealth or portfolio value)** let φ_t be any such portfolio, the wealth of an investor or the value of this portfolio at time $t, 0 \le t \le N$, is defined as

$$V_t := x_t B_t + y_t S_t \qquad (3.3)$$

Definition ***3.3*** (**Self-financing portfolio)** A trading opportunity φ_t is said self-financing if for all time $t = 1,2,\dots,N$, we have

$$Vt - Vt - 1 = x_t(Bt - Bt - 1) + yt(St - St - 1) \qquad (3.4)$$

which means that the change in the investor's wealth results only from the change in the bank account balance (or the bond price) and in the stock prices, no need to deposit/withdraw from the bank account and no need to buy/sell more shares of the stock.

Definition ***3.4*** **(Arbitrage opportunity).** An arbitrage opportunity in the multi-step binomial market model is any self-financing strategy verifying

$$V0 = 0\, and \mathbb{P}(Vt \ge 0) = 1 \quad with\ \mathbb{P}(Vt > 0) > 0;\ for\ some\ time\ t = 1,2,\dots,N \qquad (3.5)$$

Which means an investor generates riskless profit by starting with nothing and terminating with an almost sure positive wealth with strictly positive probability of strictly positive wealth at some future trading time t. We say that the market (model) is arbitrage-free if there is no arbitrage opportunity in that market (model). Absence of arbitrage principle in the multi-step binomial model is verifiable with the result below.

Proposition3.4. The multi-step binomial market model is arbitrage-free if and only $d < 1 + r < u$.

3.3 Absence of arbitrage and existence of equivalent martingale measures

Set ~ $\widetilde{S_t} := \frac{S_t}{B_t} = (1 + r)^{-t} S_t$, the discounted prices process of the stock, discounted by the bank account balance B_t at each time $t = 0, 1, \dots, N$. Clearly the natural filtration $\widetilde{S_t}$ of is also that of St. Also set $\widetilde{V_t} := (1 + r)^t\, V_t$ the discounted value process of any strategy φ_t in this multi-step binomial model.

Definition 3.5. We say that a probability measure on (Ω, F) is an equivalent martingale probability measure (usually denoted EMM) if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and the discounted stock prices process $\widetilde{S_t}$ is a $\mathbb{Q}$-martingale with respect to the natural filtration $\mathbb{F}$ of the stock prices process.

Remark3.5. If $\mathbb{Q}$ is an equivalent martingale measure, and then clearly, the discounted wealth process $\widetilde{V_t}$ of an investor using any predictable strategy φ_t is also a $\mathbb{Q}$ -martingale with respect to the filtration $\mathbb{F}$

Theorem 3.6 **(Fundamental Theorem of Asset Pricing).** The multi-step binomial financial model is arbitrage-free if and only if it's admits an equivalent martingale measure.

Corollary3.6. If $\widetilde{\mathbb{P}}$ is an equivalent martingale measure in this model, then for each time $t = 0, 1, \dots, N-1$ And for every$k = 1, 2, \dots, N-t$, we have

$$St = (1 + r)^k \, \widetilde{\mathbb{E}}(S_{t+k} / \mathcal{F}_t) \qquad (3.6)$$

Where $\widetilde{\mathbb{E}}$ denotes the expectation under $\widetilde{\mathbb{P}}$.

Proof. By applying induction on k properties of conditional expectation. For more details see [4]

Corollary3.7. The multi-step binomial model admits an equivalent martingale probability measure which is $\widetilde{\mathbb{P}} = (\widetilde{p_u}, \widetilde{p_d})$, where

$$\widetilde{p_u} = \frac{(1+r) - d}{u - d} \quad and \quad \widetilde{p_d} = \frac{u - (1+r)}{u - d} \qquad (3.7)$$

Proof. It is a straightforward use of the corollary above.

3.4 Arbitrage pricing of options in the multi-step binomial model.

Let consider a European call option in this model, i.e., a financial contract initiated at time $t = 0$ (today) which gives the holder the right (but not an obligation) to buy a share of stock at a fixed agreed price K at a future time$t = N$. Two scenarios may happen at this expiry date$N$:

- Either the stock price SN is strictly greater than K, and the European call option will be exercised and it worth $S_N - K$.

- Or the stock price S_N is less than$K$, in this case the option will not be exercised by its holder because $S_N - K$ is negative and it becomes worthless.

Hence the payoff the call option is $C_N := max(S_N - K, 0)$. since the value of such a contract is known explicitly at time $t = N$, it is naturally for investors to seek what does the options worth at prior time $t \leq N$. Hence,

Definition3.7. (Fair-price for the option). The fair-price (or market price) for this call option at an earlier time$t \leq N$, is the value C_t of the option which does not generates arbitrage opportunities in the market model. Finding any possible such price can be done by mean of replicating portfolios.

Definition .3.8. (**Replicating /hedging portfolio**). A self-financing strategy φ_t in the model is said to replicate (or hedge) the call option if its terminal value V_N equals C_N.

Proposition3.8. (Pricing principle). Under no arbitrage condition, if a European call option can be hedged by a self-financing strategy φ_t with value process V_t, then we have $\overline{C_t} = V_t$, for all time $t = 0,1, \dots, N-1$.

Proof. See a general proof in Theorem 8.1 of [4, p. 173].

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Theorem3.9. (Options pricing rule). The discounted fair prices process $\tilde{\tilde{C}}_t$ for any replicated European call option in the multi-step binomial model is a $\tilde{P}$-martingale with respect to the filtration $\mathbb{F}$, implying after some simplification that the fair prices process $\overline{C_t}$ satisfies,

$\overline{C_t} = (1 + \mathrm{r})^{-1}\tilde{\mathbb{E}}(\overline{C_{t+1}} \, / \, \mathcal{F}_t \,)$ For all time $t \; = \; 0,1, \dots, N-1$ (3.8)

Where $\tilde{P} \; = \; (\widetilde{p_u} \,, \widetilde{p_d}$ is the unique equivalent martingale measure in Corollary (3.7) and $\tilde{\mathbb{E}}$ the expectation with respect to it. In particular we have

$\overline{C_t} = (1 + \mathrm{r})^{-N+t} \;\; \tilde{\mathbb{E}} \,(\, C_N / \; \mathcal{F}_t \,)$; for all time $t = 0, 1, \dots, N$ (3.9)

Proof. This directly follows from Proposition 3.8 and Remark 3.5.

Corollary 3.10 (The Cox-Ross-Rubinstein's formula) Consider a replicable European call option with payoff $C_N \; = \; max(S_N - K, 0)$ in the multi-step binomial model. Then the fair price at time $t \; = \; 0$ of this call is given by

$$\widetilde{C_0} = (1 + r)^{-N} \sum_{k=0}^{N} \binom{N}{k} \widetilde{p_u}^{K} \widetilde{p_d}^{N-k} \max(S_0 u^k \; d^{N-k} - K, 0) \qquad (3.10)$$

Proof: Given that $C_N \; = \; max(S_N - K, 0)$ is the payoff the European call option in the multi-step binomial Model. By using Theorem 3.9 above, the discounted fair price of European call option $C_N \; = \; max(S_N - K, 0)$ at time $t \in N$ is given by

$$\overline{C_t} = (1 + \mathrm{r})^{-N+t} \;\; \tilde{\mathbb{E}} \,[\, C_N / \; \mathcal{F}_t \,]$$
$$= (1 + \mathrm{r})^{-N+t} \;\; \tilde{\mathbb{E}} \,[\, max(S_N - K, 0) \; / \; \mathcal{F}_t \,]$$

Setting $t = 0$, we get

$$\begin{aligned} \overline{C_0} &= (1 + \mathrm{r})^{-N} \;\; \tilde{\mathbb{E}} \,[\, max(S_N - K, 0) \; / \; \mathcal{F}_t \,] \\ &= (1 + \mathrm{r})^{-N} \;\; \tilde{\mathbb{E}} \,[\, max(S_N - K, 0) \;\,] \\ &= (1 + r)^{-N} \textstyle\sum_{k=0}^{N} \binom{N}{k} \widetilde{p_u}^{K} \widetilde{p_d}^{N-k} \max(S_0 u^k \; d^{N-k} - K, 0) \end{aligned}$$

using Remark 3.1 with the probability measure $\tilde{\mathbb{P}}$, as required.

IV OPTIONS PRICING BLACK-SCHOLES MODELS

4.1 Model setting and assets prices dynamics

Consider a financial market model with two assets in continuous-time trading from initial time $t = 0$ to a fixed later time $t = T > 0$:

-A **riskless** asset such as a bank account or a risk-free bond with price B_t governed by the ordinary differential equation (ODE) with initial condition

$$\begin{cases} dB_t = r_c \; B_t \\ \;\; B_0 = 1 \end{cases} \qquad (4.1)$$

where r_c is the constant continuously compounding interest rate. Solving this ODE with initial condition, we get $B_t \; = c_1 e^{r_c t}$, for all time $t \in \; [0, T]$.

- And a **risky** asset such as (share of) stock whose prices processS_t, assumed square integrable under a given probability space$(\Omega, \mathcal{F}, \mathbb{P})$, follows an Ito process as the following stochastic differential equation (SDE) with initial condition

$$\begin{cases} dS_t = \mu S_t + \sigma S_t W_t \\ S_0 = s_0 \in \mathbb{R}_+ \end{cases} \quad (4.2)$$

driven by a one-dimensional standard Brownian motion $(Wt)t \in [0, T\ ]$. The real constant μ and $\sigma\ >$ are known as drift and volatility of the stock prices respectively.

Applying $Ito's$ formula (Theorem.2.2.3) on the $C^2(\mathbb{R}_+)$ function $f(x) := log(x)$ and the Ito process S_t, we obtain the explicit solution of this stochastic differential equation SDE as

$$St\ =\ S0\ exp\ (Wt\ +\ (\mu - \sigma^2/2)t\ ;\ for\ all\ time\ \ t\ \epsilon\ [0, T] \quad (4.3)$$

4. 2 Investment strategies, arbitrage and model assumptions

Definition.4.2.**(Investment strategy and value).** In this Black-Scholes model, an investment strategy is a pair of R-valued continuous-time process $\varphi_t := (\alpha_t, \gamma_t)$, where $\alpha_t$the investor's holding in the bond and γ_t is the (random) number of units of shares s/he holds in the stock at time t. The value of such a portfolio (or investor's wealth) at time $t \in [0, T]$ is defined as

$$V_t := \alpha_t\ B_t\ + \gamma_t\ S_t$$

Definition.4.3. (i)(Self-financing strategy). An investment strategy (portfolio) φ_t is said self-financing if $dV_t\ = \alpha_t\ dB_t\ + \gamma_t\ dS_t$, i.e., the change in the investor's wealth results only from the change in the bank account and in the stock price in the market.

ii) (Arbitrage). A self-financing strategy is an arbitrage opportunity in the model if $V_0 = 0$ but $V_T\ \geq\ 0$ almost sure with $\mathbb{P}(V_T > 0) > 0$.

4.3 Black Scholes Model assumptions

As stated in this model assumes the following conditions:

1. There is no arbitrage opportunity in this market model,
2. There is no transaction cost in purchasing shares of stock,
3. The stock pays no dividend, i.e., no benefit payment to the shareholders.
4. Short-selling (i.e., borrowing and selling) is allowed in this market.
5. The market model is liquid, i.e., one can hold any real number of shares of stock.

4.4. Arbitrage pricing of call options in the Black-Scholes model.

Consider a European call option which is a contract set at time $t = 0$ (today) and which gives the holder the right (but not an obligation) to buy a share of stock at a fixed agreed price K at a future time T. Hence the payoff (at time T) of this option is $C(T, S_T) := max(S_T - K, 0)$.

As argued in the discrete-time binomial model setting, since the value of such a contract is known explicitly at time $t = T$, it is naturally for investors to seek what does the options worth at prior time$t \leq T$.

Definition .4.4.(Fair-price of an option). The fair-price at time $t \leq T$ of the call option is the price $C(t, S_t)$ that does not generate arbitrage opportunity in the model. Finding possibly such a price can also be done via replicating/hedging portfolios.

Definition .4.5.(Replicating portfolio). A self-financing strategy φ_t in the Black-Scholes model is a replicating (or hedging) strategy for a call option if its value at expiry date equals the payoff of the option, i.e., if $V_T = C(T, S_T)$.

Proposition 4.6.(Pricing principle). If a call option admits a replicating portfolio with value process V_t, then we have $C(t, S_t) = V_t$ for all time$t \leq T$.

Proposition 4.7 Under the equivalent martingale measure $\widetilde{\mathbb{P}}$, the discounted fair-prices process $C\widetilde{(t, S_t)} := e^{r_c t}\, C(t, S_t)$ is martingale with respect to $\mathbb{F}^W$.

Next, assuming that the underlying fair-price function $C(t, x)$ is in $C^{1,2}([0;\, T]\ x\ \mathbb{R}_+)$, then we have,

Theorem 4.8.(The Black-Scholes PDE). For any replicable European call option, its fair-price function $C(t, x), x \in \mathbb{R}_+$, at any prior time $t \leq T$ solves the following initial value partial differential equation (PDE),

$$\begin{cases} C_t(t,x) + \sigma^2 \frac{x^x}{2} C_{xx}(t,x) + r_c x C_x(t,x) = r_c C_x(t,x) \\ \qquad C(T,x) = \max(x - K, 0) \end{cases} \tag{4.4}$$

Where Cy and Cyy denote the partial and second partial derivatives of $C(t, x)$ with respect to $y = t$ or x.

Proof : Applying Ito's formula on the discounted fair-price process $C\widetilde{(t, S_t)}$ we obtain,

$$\mathrm{d}\widetilde{C(t,} S_t) = e^{-r_c t}\left[-r_c C(t,S_t) + C_t(t,S_t) + \frac{1}{2} S_t{}^2 \sigma^2 C_{xx}(t,S_t) + r_c S_t C_x(t,S_t)\right] dt + \left[\sigma C_x(t,S_t)\widetilde{S_t}\right] d\widetilde{W_t}$$

Under the martingale measure $\widetilde{\mathbb{P}}$, under which $\tilde{C}(t, St)$ is a martingale by the preceding Proposition 4.7. Hence the result follows by Proposition 2.2.2, along with the payoff (boundary) condition $C(T, S_T) = max(S_T - K, 0)$.

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IJARSE
ISSN: 2319-8354

Corollary.4.9. (The Black-Scholes's formula). Under the condition of the theorem 4.8. the time 0 fair-price of a replicable European call option is given by

$$C(0, S_0) = S_0\Phi(d_1) - Ke^{-r_c T}\Phi\big(d_1 - \sigma\sqrt{T}\big) \tag{4.5}$$

Where $d_1 = \frac{\log\left(\frac{S_0}{K}\right)+\left(r_c+\sigma^2/2\right)}{\sigma\sqrt{T}}$,S_0 is the current price, K is strike price, r_c is the continuous compounded risk-free rate, T is expiration date, and σ^2 is the variance of the continuously compounded return of the stock and Φ is the cumulative distribution function of the standard normal distribution.

Lemma 4.10. Let$X \sim N(\mu, \sigma)$. If a and c are positive constants, then

$$\mathbb{E}\big(max(ae^x - c, 0)\big) = ae^{\left(\mu + \frac{\sigma^2}{2}\right)}\Phi\left(\frac{\log(a/c) + \mu}{\sigma} + \sigma\right) - c\,\Phi\left(\frac{\log(a/c) + \mu}{\sigma}\right) \tag{4.7}$$

Proof. Applying Proposition.2.1.5.(4), we get

$$\mathbb{E}(\max(ae^x - c, 0)) = a\mathbb{E}\left(e^x 1_{[X > \log(c/a)]}\right) - c\mathbb{P}[X > \log(c/a)]$$

$$= ae^{\mu+\frac{\sigma^2}{2}}\,\mathbb{P}\left[X > log\left(\frac{c}{a}\right) - \sigma^2\right] - c\mathbb{P}[X > \log(c/a)]$$

$$= ae^{\mu+\frac{\sigma^2}{2}}\left(1 - \Phi\left(\frac{\log(a/c) - \mu}{\sigma} - \sigma\right)\right) - c\left(1 - \Phi\left(\frac{\log(a/c)}{\sigma} - \sigma\right)\right)$$

By using the scaling property of the normal distribution in Proposition .2.1.5.(3). Finally using the property $\Phi(-x) = 1 - \Phi(x)$ for all $x \geq 0$ from Proposition.2.1.5. (1), we get

$$\mathbb{E}\big(max(ae^x - c, 0)\big) = ae^{\left(\mu + \frac{\sigma^2}{2}\right)}\Phi\left(\frac{\log(a/c) + \mu}{\sigma} + \sigma\right) - c\,\Phi\left(\frac{\log(a/c) + \mu}{\sigma}\right)$$

Proof. (Theorem 4.8). Feynman-Kac's formula entitles from Theorem 2.2.5 that the solution of the Black-Scholes PDE (of equation 4.4) for the fair-price at any time $t \leq T$ is

$$C(t, x) = \mathbb{E}\left(e^{-(T-t)r_c}\max(S_T - x, 0)\right)$$

Hence, the result follows for time $t = 0$ by applying the preceding lemma.

4.0. Convergence of the binomial to the Black-Scholes model

In this section we discuss the convergence of the multi-step binomial model (which is discrete-time model) to the Black-Scholes model which is continuous-time model by using concept from probability theory. In fact, as exposed in [10], this convergence consists in showing that the Cox-Ross-Rubinstein (CRR) formula in Corollary 3.10

IJARSE
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converges to the Black-Scholes formula of Corollary 4.9 for a European call option. Other related literatures also read are references [4, 6, 5].

1.1 Convergence of the CRR formula to the Black-Scholes formula

In section 3, by corollary 3.10, we have seen that under Binomial model fair price at time $t = 0$ of
European call option which known as CRR formula is given by

$$\widetilde{C_0} = (1 + r_d)^{-N} \sum_{k=0}^{N} \binom{N}{k} \widetilde{p_u}^K \widetilde{p_d}^{N-k} \max(S_0 u^k\, d^{N-k} - K, 0) \tag{5.1}$$

where r_d denotes here the per step constant rate r used in the multi-step binomial model.
Moreover, in section 4 in Corollary 3.10, we have seen that in the Black-Scholes model, the fair price of a replicable European call option at time t = 0 is given by

$$C(0, S_0) = S_0 \Phi(d_1) - K e^{-r_c T} \Phi(d_2) \tag{5.2}$$

$d_1 - \sigma\sqrt{T}$ Where $d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r_c + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{K}\right) + \left(r_c - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}$, S_0 is the current price, K is strike price, r_c is the continuous compounded risk-free rate, T is expiration date, and σ^2 is the variance of the continuously compounded return of the stock and Φ is the cumulative distribution function of the standard normal distribution.

In Equation (5.1) above, the value of $\max(S_0 u^k\, d^{N-k} - K, 0)$ is zero for $a < k$, where a is the minimum number of upward movement for the call option finish the money. That means for smallest integer $(a \le N)$ such that$S_0 u^a\, d^{N-a} > K$.

For $k < a$, we have $\max(S_0 u^k\, d^{N-k} - K, 0) = 0$ and for $a \le K, S_0 u^k\, d^{N-k} > K$, we have $\max(S_0 u^k\, d^{N-k} -$ *K,0= S0uk dN−k−K*. Now we need to count the binomial path from $K = a$ to N, hence the equation (5.1) becomes

$$\widetilde{C_0} = (1 + r_d)^{-N} \sum_{k=a}^{N} \binom{N}{k} \widetilde{p_u}^K \widetilde{p_d}^{N-k} \; [S_0 u^k\, d^{N-k} - K] \tag{5.3}$$

Remark 5.2. Using equation (5.3), we can write fair price at time $t = 0$ of European call option in Binomial model which known as CRR formula as follows compactly

$$\overline{C_0} = S_0 B_1 - (1 + r_d)^{-N} K B_2 \tag{5.4}$$

Where

$$B_1 = \sum_{k=a}^{N} \left[\left(\frac{u}{1 + r_d}\right) \widetilde{p_u}\right]^K \left[\left(\frac{d}{1 + r_d}\right) \widetilde{p_d}\right]^{N-K} = \sum_{k=a}^{N} \widetilde{p_u^*}^K \widetilde{p_d^*}^{N-K} \tag{5.5}$$

With $\widetilde{p_u}^* = \frac{u}{1 + r_d}$, $\widetilde{p_d}^* = 1 - \widetilde{p_u}^*$ And

IJARSE
ISSN: 2319-8354

$$B_2 = \sum_{k=a}^{N} \widetilde{P_u}^{K} \widetilde{P_d}^{N-K} \tag{5.6}$$

Lemma 5.3 The call option formula in the given in equation (5.4) is equivalent to

$$C_0 = S_0 B_1 - Ke^{-rcT} B_2 \tag{5.7}$$

Proof. First note that $(1+r_d)^{-N}$ is the present value factor for N periods. The per period rate r_d can be related to an annual rate r_a applied for T years by the relationship $r_d = r_a^{\frac{1}{N_a}}$ where N_a a is the number of periods per year. Hence,

$$1 + r_d = (1+r_a)^{\frac{1}{N_a}} \Rightarrow (1+r_d)^N = (1+r_a)^{\frac{N}{N_a}} = (1+r_a)^T \quad \text{Where} \quad T = \frac{N}{N_a} \tag{5.8}$$

Since the present value factor for T years is$(1+r_a)^T$, then the present value denoted PV of 1 Dollar is

$$PV = (1+r_a)^{-T} \Rightarrow \log(PV) = \log((1+r_a)^{-T}) = -\mathrm{T}\log(1+r_a)$$

$$PV = e^{-\mathrm{T}\log(1+r_a)}$$

Hence, setting $r_c = log(1+r_a)$, the present value factor $((1+r_d)^N$ in the multi-step binomial model is equivalent to the present value factor $e^{-r_c T}$ when continuously compounded interest rate r_c applies. It follows that the call option formula in the given in equation (5.4) is equivalent to

$$C_0 = S_0 B_1 - Ke^{-rcT} B_2$$

Lemma 5.4. The minimum number of movements for call option finish the money a is given by

$$a = \frac{log\left(\frac{K}{S_0}\right) - Nlog(d)}{log\left(\frac{u}{d}\right)} + \gamma \tag{5.9}$$

Where $\gamma \in (0,1)$ which is the number added to a to make integer.

Proof. For$a \le k$, we required that $S_0 u^a d^{N-a} > K$ and by applying property of logarithm and doing some calculation, we have

$$a > \frac{log\left(\frac{K}{S_0}\right) - Nlog(d)}{log\left(\frac{u}{d}\right)} \tag{5.10}$$

When we see carefully equation (5.10), it is not integer. So to express a as integer we have to add to it a number $\gamma \in (0,1)$

Lemma 5.5 The binomial distribution of B1 converges to the normal distribution $\phi(x)$ which is given by

$$\int_{-\infty}^{x} \phi(t)dt \tag{5.11}$$

Proof. De Moivre-Laplace Central Limit Theorem2.1.10. says that a binomial distribution converges to the normal distribution if $N\widetilde{p_u} \to \infty$ as$N \to \infty$. In our case $B_1 \to \int_a^{\infty} \phi(k)dk$ where $\phi(k)$the density distribution function for

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IJARSE
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normal distribution. But, k is not standard normal random variable. So, let us convert k into standard normal random variable by defining $x = \frac{k-\mathbb{E}(k)}{\sigma_k}$. Then, we have

$$\int_{a}^{\infty} \phi(k)dk = \int_{x}^{\infty} \phi(t)dt \quad where\ x = \frac{k-\mathbb{E}(k)}{\sigma_k} \tag{5.12}$$

Let $x = -\frac{k-\mathbb{E}(k)}{\sigma_k}$, then equation (5.12) become

$$\int_{-\infty}^{x} \phi(t)dt = \Phi(t)$$

Remark.5.6. By applying the same concept as $B_1 \rightarrow \Phi(\mathrm{x})$, then B_2 also converges to normal distribution $\Phi(x)$.

Lemma.5.7. Let S_T be the stock price at expiration date T. After N period of time and k up wards movements in the markets, $S_T = S_0 u^k d^{N-k}$, then expectation and variance of k are given by

$$\mathbb{E}(K) = \frac{\mathbb{E}\left(log\left(\frac{S_T}{S_0}\right) - Nlog(d)\right)}{log\left(\frac{u}{d}\right)} \quad and \quad \mathbb{V}ar(k) = \frac{\mathbb{V}ar\left(log\left(\frac{S_T}{S_0}\right)\right)}{\left(log\left(\frac{u}{d}\right)\right)^2} \tag{5.13}$$

Proof. Straightforward

Remark.5.8. i) Using lemma (5.4) and lemma (5.7), and then as $N \rightarrow \infty$ the value of x become

$$x = \frac{log\left(\frac{S_0}{k}\right) + \mathbb{E}\left(log\left(\frac{S_T}{S_0}\right)\right)}{\sqrt{\mathbb{V}ar\left(log\left(\frac{S_T}{S_0}\right)\right)}} \tag{5.14}$$

ii) Since our discrete binomial process converge to continuous log-normal process, and then we have $\sqrt{\mathbb{V}ar\left(log\left(\frac{S_T}{S_0}\right)\right)} = \sigma^2 T$. Using this result, and substituting in equation (5.14), we have

$$x = \frac{log\left(\frac{S_0}{k}\right) + \mathbb{E}\left(log\left(\frac{S_T}{S_0}\right)\right)}{\sigma^2 T} \tag{5.15}$$

Theorem.5.9. The call option price formula in the binomial model written as in Lemma (5.3) converges to call option price formula in the Black-Scholes model given in equation 5.7. In particular B_1 and B_2 converge to $\Phi(d_1)$ and $\Phi(d_2)$ respectively.

IJARSE
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Proof. We need the value of x in equation (5.15) to equal d_1 and d_2 as defined by the Black-Scholes formula with the probabilities are $\widetilde{p_u}^*$ and $\widetilde{p_u}$. That means we need to verify

$$\mathbb{E}\left(log\left(\frac{S_T}{S_0}\right)\right) = \left(r_c + \frac{\sigma^2}{2}\right)T, \qquad if\ the\ probability\ is\ \widetilde{p_u}^*$$

$$\mathbb{E}\left(log\left(\frac{S_T}{S_0}\right)\right) = \left(r_c + \frac{\sigma^2}{2}\right)T, \ \ if\ the\ probability\ is\ \widetilde{p_u}$$

Let verify one by one. From remark (5.2), we have

$$\widetilde{p_u}^* = \frac{u}{1+r_d}\widetilde{p_u} = \frac{u}{1+r_d}\left(\frac{r_d - d}{u-d}\right) \quad \Rightarrow\ r_d = \left[\widetilde{p_u}^*\left(\frac{1}{u}\right) + (1-\widetilde{p_u}^*)\left(\frac{1}{d}\right)\right]^{-1}$$

Then, using equation (5.7), we have

$$(1+r_d)^N = \left[\widetilde{p_u}^*\left(\frac{1}{u}\right) + (1-\widetilde{p_u}^*)\left(\frac{1}{d}\right)\right]^{-N} = \left[\widetilde{p_u}^*\left(\frac{1}{u}\right) + \widetilde{p_d}^*\left(\frac{1}{d}\right)\right]^{-N} = (1+r_a)^T \qquad (5.16)$$

Note that we can express $\frac{S_0}{S_T}$ by the following sequence,

$$\frac{S_0}{S_T} = \left(\frac{S_0}{S_1}\right)\left(\frac{S_1}{S_2}\right)\left(\frac{S_2}{S_3}\right), \dots, \left(\frac{S_{N-2}}{S_{S_{N-1}}}\right)\left(\frac{S_{N-1}}{S_T}\right) = \prod_i^N \left(\frac{S_{i-1}}{S_i}\right) \qquad (5.17)$$

Then, using equation 4.2.16, the expectation of $\frac{S_0}{S_T}$ is given by

$$\mathbb{E}\left(\frac{S_0}{S_T}\right) = \mathbb{E}\left[\prod_i^N \left(\frac{S_{i-1}}{S_i}\right)\right] = \prod_i^N \mathbb{E}\left(\frac{S_{i-1}}{S_i}\right) \qquad (5.18)$$

From lemma (5.3), the probability for B_1 is $\widetilde{p_u}^*$. Since$S_i = S_{i-1}u$ with probability $\widetilde{p_u}^*$ and $S_i = S_{i-1}d$ with probability $\widetilde{p_d}^*$. Hence, we have

$$\mathbb{E}\left(\frac{S_0}{S_T}\right) = \widetilde{p_u}^*\left(\frac{1}{u}\right) + \widetilde{p_d}^*\left(\frac{1}{d}\right) \qquad (5.19)$$

When we substitute equation (5.18) into (5.19), we have

$$\mathbb{E}\left(\frac{S_0}{S_T}\right) = \prod_i^N \left[\widetilde{p_u}^*\left(\frac{1}{u}\right) + \widetilde{p_d}^*\left(\frac{1}{d}\right)\right] = \left[\widetilde{p_u}^*\left(\frac{1}{u}\right) + \widetilde{p_d}^*\left(\frac{1}{d}\right)\right]^N = (1+r_a)^T$$

By using equation (5.16)

This implies,

$$-Tlog(1+r_a) = log\left(\mathbb{E}\left(\frac{S_0}{S_T}\right)\right) = \mathbb{E}\left(log\left(\frac{S_0}{S_T}\right)\right) \qquad (5.20)$$

Since $\frac{S_T}{S_0}$ log-normally distributed, its inverse also log-normally distributed. Hence, $\frac{S_0}{S_T}$ is also log-normally distributed. By using the $\mathbb{E}(x) = exp\left(\mu + \frac{\sigma^2}{2}\right)$, then we have

$$log\left(\mathbb{E}(x)\right) = \mu + \frac{\sigma^2}{2} \tag{5.21}$$

By using the relation in equation (5.21), equation (5,20) and using the property of logarithm and expectation, we have

$$\mathbb{E}\left(log\left(\frac{S_T}{S_0}\right)\right) = Tlog(r) + \frac{\left[\mathbb{V}ar\left[log\left(\frac{S_T}{S_0}\right)\right]\right]^2}{2} \tag{5.22}$$

Since $log(1 + r_a) = r_c$ and $\mathbb{V}ar\left[log\left(\frac{S_T}{S_0}\right)\right] = \sigma^2 T$ equation (5.22) becomes

$$\mathbb{E}\left(log\left(\frac{S_T}{S_0}\right)\right) = r_c T + \frac{\sigma^2 T}{2} \tag{5.23}$$

By substituting equation (5.22) into equation (5.15), we have

$$x = \frac{log\left(\frac{S_0}{k}\right) + \left(r_c + \frac{\sigma^2}{2}\right)}{\sigma^2\sqrt{T}} \tag{5.24}$$

Hence, using Lemma (5.5), B_1 converges to $\Phi(x) = \Phi(d_1)$ where x is given by equation (5.24).

We can show the convergence of B_2 to $\Phi(d_1)$ by adopting the same procedure. Therefore, the call option pricing formula in the multi-step binomial model converges to call option pricing formula in the Black-Scholes model

V CONCLUSION

We introduced some useful preliminary concept from probability theory and basic concept of stochastic calculus which helped us as background of the paper. Later, We introduced the two common option pricing: multi-step binomial model and Black-Scholes model as discrete-time and continuous-time model respectively by using probability theory and basic stochastic calculus.

At the end, we introduced the main purpose of the paper which is the convergence of binomial model to Black-Scholes model. This is done by using de Moivre Laplace central limit theorem from probability theory, Cox-Ross-Rubinstein European call option formula in multi-step binomial model as discrete-time model converge to Black-Scholes pricing formula for European call option in Black-Scholes as continuous-time model.

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