Generalized Leibnitz Rule And Summation Involving I-Function of One And Two Variables

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ABSTRACT

An alternative proof of the generalized Leibnitz rule is attempted; A slightly generalized form of this rule is applied in deriving a certain series summation believed to be new, involving the product of two I-functions given by Saxena V.P. in 1982 [11] and its corresponding integral analogue obtained, the results yield various series summation and their corresponding integral analogues as given by Osler in (1971) [4].

Keywords: I-function of one and two variables, Leibnitz rule, known integrals, etc.

I.INTRODUCTION

The Leibnitz rule of the elementary calculus given in the following form,

$$D_z^n u.v=\sum_{n=0}^{\infty} {}^N C_n . D_z^{N-n} u. D_z^n v$$

(1.1)

has its generalization given by Osler[4] in (1972),

$$D_z^{\alpha} u.v=\sum_{n=0}^{\infty} {}^{\alpha} C_n . D_z^{\alpha-n} u. D_z^n v$$

(1.2)

where D_z^{α} denotes the fractional derivative operator which is an extension of the familiar derivative operator $D_z^{\alpha} = \frac{d^n}{dz^n}$ to arbitrary (real or complex) order. As pointed out by Osler equation (1.2) has distributing feature, in the sense that if the functions u and v are interchanged, L.H.S. remains unchanged, the R.H.S. is not so, this is clear for the fact that whole 'u' is differentiated fractionally and 'v' is differentiated in the usual elementary sense.

Osler (1972) has further generalized the Leibnitz rule (1.2). Various forms in which the interchanging of the functions 'u' and 'v' appears to permissible. One of his generalizations is:

$$D_z^{\alpha}u(z).v(z)=\sum_{r=-\infty}^{\infty}\binom{\alpha}{r+\gamma}.D_z^{\alpha\text{-r-}\gamma}u(z).D_z^{r+\gamma}v(z),$$

(1.3)

valid for all α and γ.

In this paper, we attempt to prove (1.3) in such a manner which is different from that given by Osler in 1971. A slightly generalized form of (1.3) is, then used to derive a series by Osler [4]. A slightly generalized form of (1.3) is then used to derive a series summation involving the product of I-function of one variable given by Saxena, V.P. in 1982 [11]. Lastly the concluding section gives the integral analogue of the results derived in section 3.

The I-function occurring in this paper are the well known I-function of Saxena, V.P. (1981)[11] and its generalization to two variables, these functions are contour integrals of double Mellin-Barnes type given by Shanta Kumari,K. *et.al.*[12].

For the notation, definition, condition of existence and other important properties of these function, given in the papers of Saxena, V.P.[11], Agarwal *et.al.*[1].and Shanta Kumari,K. *et.al.* 2013 [12] are reference too.

1. Generalized Leibnitz rule :

Let **R** be a simple connected open region containing $z=0$, and let p be the largest real number such that domain $|z|<p$ is entirely contained in **R**.

Let $u(z)=(az)^{p}f(z)$ and $v(z)=(bz)^{q}g(z)$

(2.1)

$$\text{Re}(p)>-1,\quad \text{Re}(q)>-1,$$

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Then for all α and γ such that $\binom{\alpha}{\gamma+r}$ is defined by ${}^{\alpha}C_{\gamma+r}$ where $\alpha \geq \gamma + r$.

The left hand member L (say) of (2.1) then lead to,

$$\mathrm{L}=\mathrm{D}_{\mathrm{z}}^{\alpha}\sum_{\mathrm{m}=0}^{\infty}\sum_{\mathrm{n}=0}^{\infty}A_m B_n Z^{p+q+m+n}$$

$$=\sum_{\mathrm{m}=0}^{\infty}\sum_{\mathrm{n}=0}^{\infty}A_m B_n D_{\mathrm{z}}^{\alpha} Z^{p+q+m+n}$$

(2.2)

The interchanging of the fractional derivative operator and the summation is justifiable in view of the condition mentioned with (2.2).

Since, $D_{\mathrm{z}}^{\alpha}(az)^{\rho} = \dfrac{a^{\rho}\Gamma(\rho+1)}{\Gamma(\rho+\alpha+1)}, \mathrm{Re}(\rho) > -1,$

(2.3)

Which held for all values of α, then L reduces to

$$\mathrm{L}=\sum_{\mathrm{m}=0}^{\infty}\sum_{\mathrm{n}=0}^{\infty}A_m B_n \frac{\Gamma(p+\mathrm{q}+\mathrm{m}+n+1)}{\Gamma(p+\mathrm{q}+\mathrm{m}+n-\alpha+1)}\langle a\rangle^{p+m}\langle b\rangle^{q+n} Z^{p+q+m+n-\alpha},$$

(2.4)

Turning now to the right hand member R (say), we now applying (2.3), see research paper of Raina [7],

$$R \equiv \sum_{r=-\infty}^{\infty}\binom{\alpha}{r+\gamma} D_{\mathrm{z}}^{\alpha-r-\gamma}\left(\sum_{m=0}^{\infty}A_m (az)^{p+m}\right) D_{\mathrm{z}}^{r+\gamma}\left(\sum_{n=0}^{\infty}B_n (\mathrm{b}\,z)^{q+n}\right)$$

$$= \sum_{r=-\infty}^{\infty}\binom{\alpha}{r+\gamma}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_m B_n \langle a\rangle^{p+m}\langle b\rangle^{q+n}.$$

$$\frac{\Gamma(p+m+1)\Gamma(\mathrm{q}+n+1)\,\mathrm{z}^{p+q+m+n-\alpha}}{\Gamma(\mathrm{p}+m+r+\gamma-\alpha+1)\Gamma(\mathrm{q}+n-r-\gamma+1)},$$

(2.5)

Assuming that the order of the summations can be changed, we can write,

$$\mathrm{D}_{\mathrm{z}}^{\alpha}\left(\mathrm{u(z).v(z)}\right)=\sum_{\mathrm{r}=-\infty}^{\infty}\binom{\alpha}{r+\gamma}.\mathrm{D}_{\mathrm{z}}^{\alpha\text{-r-}\gamma}\mathrm{u(z)}.\mathrm{D}_{\mathrm{z}}^{r+\gamma}\mathrm{v(z)},$$

(2.6)

Where $Re(p+q) > -1$, and $0 < |z| < p$.

Proof: As the functions $f(z)$ and $g(z)$ are analytic in the circuler domain $|z| < p$, let their power series expansions about origin will be given by,

$$f(z)=\sum_{m=0}^{\infty} A_m (az)^m \quad \text{and} \quad g(z)=\sum_{n=0}^{\infty} B_n (\mathrm{b}\, z)^n \tag{2.7}$$

respectively.

$$R=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a)^{p+m}(b)^{q+n} A_m B_n \times$$

$$\sum_{r=-\infty}^{\infty}\binom{\alpha}{r+\gamma}\frac{\Gamma(p+m+1)\Gamma(\mathrm{q}+n+1)\Gamma(\alpha+1)\,\mathrm{z}^{p+q+m+n-\alpha}}{\Gamma(\mathrm{p}+m+r+\gamma-\alpha+1)\Gamma(\mathrm{q}+n-r-\gamma+1)\Gamma(1+\alpha-\gamma)\Gamma(1+\gamma)}\times$$

$${}_2\mathrm{H}_2\left[\gamma-\alpha;\gamma-q-n;1+\gamma-\alpha-p+\mathrm{m},1+\gamma;1\right], \tag{2.8}$$

where the function ${}_2\mathrm{H}_2$ denoted a bilateral series using the generalized Gauss theorem (1966) viz.

$${}_2\mathrm{H}_2\left[\mathrm{a},\mathrm{b};\mathrm{c};\mathrm{d};1\right]=\frac{\Gamma(c)\Gamma(\mathrm{d})\Gamma(1-a)\Gamma(1-b)\Gamma(\mathrm{c}+\mathrm{d}-a-b-1)}{\Gamma(\mathrm{c}-a)\Gamma(\mathrm{d}-a)\Gamma(\mathrm{c}-b)\Gamma(\mathrm{d}-b)}, \tag{2.9}$$

The right hand member of (2.8) after a little simplification yields the left hand member L given in (2.2). This proves (2.2).

II.SUMMATION

In this section we evaluate summation of series involving product of I-function of one and two variables, for that we derive the following results:

$$\sum_{r=-\infty}^{\infty} a\binom{\alpha}{ar+\gamma}\mathrm{I}^{m,n+1}_{p_i+1,\mathrm{q}_i+1;r}\left[cz^h\left|\begin{matrix}(-\lambda;h),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1},\mathrm{p}_r\\(\mathrm{b}_j,\beta_j)_{1,\mathrm{m}};(\mathrm{b}_{ji},\beta_{ji})_{\mathrm{m}+1,\mathrm{q}_i};(-\lambda+\alpha-ar-\gamma;h)\end{matrix}\right.\right]\times$$

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$$I^{M,N+1}_{P_i+1,Q_i+1;r}\left[dz^k \left| \begin{matrix} (-\mu;k),(a_j',\alpha_j')_{1,N};(a_{ji}',\alpha_{ji}')_{N+1},P_r \\ (b_j',\beta_j')_{1,M};(b_{ji}',\beta_{ji}')_{M+1,Q_i};(-\mu+ar+\gamma;k) \end{matrix}\right.\right]$$

$$=I^{0,1;m,n;M,N}_{1,1;p_i,q_i,P_i,Q_i}\left[\begin{matrix} cz^h \\ dz^k \end{matrix}\left| \begin{matrix} (-\lambda-\mu;h,k;1),(a_j;\alpha_j,A_j;\xi_j)_{1,p_1};(c_j,C_j;U_j)_{1,p_2},(e_j,E_j;P_j)_{1,p_3} \\ (-\lambda-\mu+\alpha;h,k;1),(b_j;\beta_j,B_j;\eta_j)_{1,q_1};(d_j,D_j;V_j)_{1,q_2},(f_j,F_j;Q_j)_{1,q_3} \end{matrix}\right.\right],$$

where $h>0,\ k>0, 0<a\le 1,\ \ Re\left[\lambda+h\left(\frac{b_j}{\beta_j}\right)\right]>-1,\ \ Re\left[\mu+k\left(\frac{b_j}{\beta_j}\right)\right]>-1,$ **(3.1)**

(i=1,2,…..m), (j=1,2,…….,M) and for I-function of two variables[12],

$$Re\left[\lambda+h\left(\frac{d_jV_j}{D_j}\right)+h\left(\frac{f_jQ_j}{F_j}\right)\right]>-1.$$

$$|\arg\ c|<\frac{1}{2}\left[\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p_i}\alpha_{ji}+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q_i}\beta_{ji}\right]\pi, \quad \textbf{(3.2)}$$

$$|\arg\ d|<\frac{1}{2}\left[\sum_{j=1}^{N}\alpha_j'-\sum_{j=N+1}^{P_i}\alpha_{ji}'+\sum_{j=1}^{M}\beta_j'-\sum_{j=M+1}^{Q_i}\beta_{ji}'\right]\pi,$$

(3.3)

Proof of (3.1): A slightly generalized form (2.2) given by Osler (1972) is,

$$D^{\alpha}_{\phi(z)}u(z).v(z)=\sum_{r=-\infty}^{\infty}\binom{\alpha}{ar+\gamma}.D^{\alpha-ar-1}_{\phi(z)}u(z).D^{ar+\gamma}_{\phi(z)}v(z), \quad \textbf{(3.4)}$$

where the fractional derivative occurs with respect to the arbitrary function $\phi(z)$ and the sum taken over a times r, $(0\le a<1)$.

Setting

$$u(z)=z^{\lambda}\mathrm{I}_{p_i,q_i;r}^{m,n}\left[cz^{h}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right],$$

(3.5)

$$v(z)=z^{\mu}\mathrm{I}_{P_i,Q_i;r}^{M,N}\left[dz^{k}\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right], \quad \textbf{(3.6)}$$

Then

$$D_z^{\alpha}\left\{z^{\lambda+\mu}\mathrm{I}_{p_i,q_i;r}^{m,n}\left[cz^{h}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right].\mathrm{I}_{P_i,Q_i;r}^{M,N}\left[dz^{k}\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]\right\}$$

$$=\sum_{r=-\infty}^{\infty}a\binom{\alpha}{ar+\gamma}.D_z^{\alpha\text{-}ar\text{-}\gamma}z^{\lambda}\mathrm{I}_{p_i,q_i;r}^{m,n}\left[cz^{h}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\times$$

$$D_z^{ar+\gamma}z^{\mu}\mathrm{I}_{P_i,Q_i;r}^{M,N}\left[dz^{k}\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right],$$

(3.7)

Using the known results giving the fractional derivatives of I-function due to Sharma (2018) [10]

$$D_z^{q}z^{\lambda}\mathrm{I}_{P_i,Q_i;r}^{M,N}\left[cz^{h}\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$$=z^{\lambda-q}\mathrm{I}_{P_i+1,Q_i+1;r}^{M,N+1}\left[cz^{h}\left|\begin{matrix}(-\lambda,h),(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i},(-\lambda+q,h)\end{matrix}\right.\right]$$

Provided $Re\left[\lambda+h\left(\frac{b_{ji}}{\beta_{ji}}\right)\right]>-1, (i=1,2,\dots..r)$ and $(j=1,2,\dots\dots,M)$ and $h>0$, and

$$D_z^q z^{\lambda} I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|c} az^h & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1};(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ bz^k & (b_j;\beta_j,B_j;\eta_j)_{1,q_1};(d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{array}\right]$$

$$= z^{\lambda-q} I^{0,n_1+1:m_2,n_2;m_3,n_3}_{p_1+1,q_1+1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|c} az^h & (-\lambda;h,k;1),\ (a_j;\alpha_j,A_j;\xi_j)_{1,p_1};(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ bz^k & (b_j;\beta_j,B_j;\eta_j)_{1,q_1};(-\lambda+q;h,k;1);(d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{array}\right],$$

valid for all values of q provided that

$Re\left[\lambda + h\left(\frac{d_jV_j}{D_j}\right) + k\left(\frac{f_jQ_j}{F_j}\right)\right] > -1, (i = 1,2,\ldots.m_2)$ and $(j = 1,2,\ldots\ldots,n_2)$ and $h > 0, k > 0$.

(3.8)

It being assumed that the I-function of one variable [11] in (1982) and two variables [12] in (2013) satisfied the condition of existence mentioned in Shantha Kumari,K, Vasudevan Nambisan T.M., and Rathi, A.K.(2013) [12] and Mathai, A.M., Saxena R.K. and Haubold, H.J. in (2009)[6], respectively.

The standard results can be easily arrived from particular cases of (3.1).

For instance if all the exponents $\xi_j(j=1,\ldots\ldots,p_1), \eta_j(j=1,\ldots\ldots,q_1), U_j(j=1,\ldots\ldots,p_2), V_j(j=1,\ldots\ldots,q_2)$, $P_j(j=1,\ldots\ldots,p_3), Q_j(j=1,\ldots\ldots,q_3)$ are equal to unity the (3.1) reduces to the H-function of two variables defined by Mittal and Gupta [3] and reduced result given by Raina, R.K. in (1978) [7].

Similarly other relations may be obtained.

III.INTEGRAL ANALOGUE

Osler (1972) in another paper has given the integral analogue of the generalized Leibnitz rule (2.6) as follows:

$$D^{\alpha}_{\phi(z)} u(z).v(z) = \int_{-\infty}^{\infty} \binom{\alpha}{\omega+\gamma} D^{\alpha-\omega r}_{\phi(z)} u(z).D^{\omega+\gamma}_{\phi(z)} v(z)\, d\omega, \qquad \textbf{(4.1)}$$

Where α and γ are arbitrary (real or complex).

Application of (4.1) for the specified function $u(z)$ and $v(z)$ in terms of I-function of one variable (3.5) and (3.6) automatic yield the following integral analogue of (3.1),

$$\int_{-\infty}^{\infty}\binom{\alpha}{\omega+\gamma} I^{m,n+1}_{p_i+1,q_i+1;r}\left[cz^h\left|\begin{matrix}(-\lambda,h),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(-\lambda+\alpha-\omega,h)\end{matrix}\right.\right]$$

$$\times I^{M,N+1}_{P_i+1,Q_i+1;r}\left[dz^k\left|\begin{matrix}(-\mu,k),(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i},(-\mu+\gamma+\omega,k)\end{matrix}\right.\right]d\omega$$

$$=I^{0,1;m,n;M,N}_{1,0;p_i,q_i;P_i,Q_i}\left[\begin{matrix}cz^h\\dz^k\end{matrix}\left|\begin{matrix}(-\lambda-\mu;h,k,1);(a_j;\alpha_j,A_j;\xi_j)_{0,1};(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3}\\(-\lambda-\mu+\alpha;h,k,1);(b_j;\beta_j,B_j;\eta_j)_{1,0};(d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{matrix}\right.\right], \tag{4.2}$$

Provided conditions mentioned with (3.1) to (3.3) satisfied.

IV.CONCLUSION

The I-function of two variables, presented in this paper, is quite basic in nature. Therefore on specializing the parameters of the function; we may obtain various other special functions such as Fox's H-function, Meijer's G-function, Wright's generalized hypergeometric function, Macrobert's E-function, generalized hypergeometric function etc. as its special cases, and therefore various unified summation presentation can be obtained as special cases of our results.

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