(0,1)'INCOMPLETE' PÁL TYPE BIRKHOFF INTERPOLATION PROBLEMS

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ABSTRACT

In this paper regularity of five (0,1) "incomplete"Pál type Birkhoffinterpolation problems involving the Möbius transform of the zeros ofz^n+1 *and* z^n-1*has been studied.*

Keywords- Lacunary interpolation,Möbius transform,Pál type Birkhoff interpolation, Roots of unity, Regularity.

AMS Subject Classification: 41A05

I.INTRODUCTION

Let $\pi_n = \{P(z) \in C(z),\ degree\ of\ P(z) \leq n\}$ be the set of polynomials of degree less than or equal to n with complex coefficients. Let $A(z) \in \pi_n$ and $B(z) \in \pi_m$ then for a given positive integer r the problem of $(0,r)$Pál type Birkhoff interpolation on the pair $\{A(z), B(z)\}$ is to find a polynomial $P(z) \in \pi_{n+m-1}$ which assumes arbitrary prescribed values at the zeros of $A(z)$ and arbitrary prescribed values of the r^{th}derivative at the zeros of $B(z)$.The problem is regular if and only if any $P(z)$ satisfying the corresponding homogeneous system of equations

$P(y_i) = 0$ Where $A(y_i) = 0\ i = 1,2,\dots,n$

$P^r(z_j) = 0$ Where $B(z_j) = 0\ j = 1,2,\dots,m$

vanishes identically. Here the zeros of $A(z)$, $B(z)$ are assumed to be simple. The problem is known as Hermite-Birkhoff interpolation if $A(z) \equiv B(z)$ [1].

Lacunary interpolation appears whenever observation gives scattered, irregular information about a function and itderivatives.Hermite-Birkhoff interpolation on real nodes is a well-developed theory[1].P. Turán [2] initiated the problem of (0,2) interpolation on the zeros of $(1-x^2)P'_{n-1}(x)$, where $P_{n-1}(x)$ is the Legendre Polynomial of degree $n-1$. O. Kiŝ [3] solved the problem when the nodes are roots of unity.Regularity of some Lacunary interpolation problems on non-uniformly distributed nodes on the unit circle has been studied. For the extensive study on the subject of Lacunary interpolation we refer [4] - [9]. R. Brueck [10] considered non-uniformly distributed nodes on the unit disk obtained by applying Möbius transform

$T_\alpha(z) = \dfrac{z-\alpha}{1-\alpha z} 0 < \alpha < 1$

to the set $U_n$of the zeros of $z^n - 1$ and $U_n'$of the zeros of$z^n + 1$. The sets $T_\alpha(U_n)$ &$T_\alpha(U_n')$ are the sets of zeros of the polynomials defined by

$v_n^{(\alpha)}(z) = (z+\alpha)^n - (1+\alpha z)^n$(1.1)

$w_n^{(\alpha)}(z) = (z+\alpha)^n + (1+\alpha z)^n$(1.2)

M.G. de Bruin and H.P. Dikshit, studied certain cases of Pál type interpolation involving the zeros of the polynomials given by (1.2) and (1.3)[11]-[13]. M.G.de Bruin[14] studied Pál type interpolation problem for nine different pair of the zeros of polynomials given by (1.2) and (1.3), where one or two of the zeros of$w_n^{(\alpha)}(z)$ and/or$v_n^{(\alpha)}(z)$ is omitted from the set of interpolation points. Such kind of problem is quite different from the problem, where one or twozeros are added to the set of interpolation points. He omitted the zeros $z = \pm 1$ from $v_n^{(\alpha)}(z)$ and/or $z = -1$ from$w_n^{(\alpha)}(z)$ and summed up all results as 'incomplete' Pál type interpolation problem.

Before our investigation, we mention a result on 'incomplete' Pál type interpolation, which is more relevant to the present study:

Theorem 1.1[14]Let$w_n^{(\alpha)}(z)$, $v_n^{(\alpha)}(z)$be as in (1.1), (1.2) with $0 < \alpha < 1$ and$n \geq 2$, then the Pál type interpolation problem is regular for the following pairs of functions:

1. $\left\{w_{n+1}^{\alpha}(z), \frac{v_n^{\alpha}(z)}{z-1}\right\}$
2. $\left\{w_n^{(\alpha)}(z), \frac{v_n^{(\alpha)}(z)}{z-1}\right\}$
3. $\left\{\frac{v_{n+1}^{(\alpha)}(z)}{z-1}, w_n^{(\alpha)}(z)\right\}$
4. $\left\{w_{2n+1}^{(\alpha)}(z), \frac{v_{2n}^{(\alpha)}(z)}{z+1}\right\}$
5. $\left\{\frac{w_{2n+1}^{(\alpha)}(z)}{z+1}, v_{2n}^{(\alpha)}(z)\right\}$
6. $\left\{\frac{v_{2n}^{(\alpha)}(z)}{z+1}, w_{2n-1}^{(\alpha)}(z)\right\}$
7. $\left\{w_{2n}^{(\alpha)}(z), \frac{v_{2n}^{(\alpha)}(z)}{z^2-1}\right\}$
8. $\left\{\frac{w_{2n+1}^{(\alpha)}(z)}{z+1}, \frac{v_{2n}^{(\alpha)}(z)}{z+1}\right\}$
9. $\left\{\frac{w_{2n+1}^{(\alpha)}(z)}{z+1}, \frac{v_{2n}^{(\alpha)}(z)}{z^2-1}\right\}$

In each case the zeros of the first polynomial of the pair are used to interpolate the value $P(.)$ and the zeros of the second one for the derivative $P'(.)$.

II.MAIN RESULTS

M.G. de Bruin[14] omitted the zeros $z = \pm 1$ from $v_n^{(\alpha)}(z)$ and/or $z = -1$ from $w_n^{(\alpha)}(z)$, while we omit the zeros$z = \pm 1$ from $v_n^{(\alpha)}(z)$ and/or $z = \zeta$ from $w_n^{(\alpha)}(z)$.

Theorem2.1: Let $w_n^{(\alpha)}(z)$, $v_n^{(\alpha)}(z)$ be defined as in (1.1), (1.2) with $0 < \alpha < 1$ and $n \geq 2$, then the (0,1) Pál type Birkhoff interpolation problem for$\left\{\frac{w_{n+1}^{(\alpha)}(z)}{z-\zeta},\ v_n^{(\alpha)}(z)\right\}$is regular.

Proof: Here we have total $n + 1 - 1 + n = 2n$ interpolation points.

The problem is to find a polynomial $P(z) \in \pi_{2n-1}$ with

$P(w_i) = 0$ Where w_i is a zero of $\frac{w_{n+1}^{(\alpha)}(z)}{z-\zeta}$ (2.1)

$P'(v_j) = 0$ Where v_j is a zero of $v_n^{(\alpha)}(z)$(2.2)

Let us take $P(z) = \frac{w_{n+1}^{(\alpha)}(z)}{z-\zeta} Q(z)$, $Q(z) \in \pi_{n-1}$(2.3)

Thus $P(z) \in \pi_{2n-1}$.

The problem will be regular if $P(z) \equiv 0$.

Equation (2.3) gives,

$$P'(z) = \frac{w_{n+1}^{(\alpha)}(z)}{z-\zeta} Q'(z) + \left\{\frac{[w_{n+1}^{(\alpha)}(z)]'}{z-\zeta} - \frac{w_{n+1}^{(\alpha)}(z)}{(z-\zeta)^2}\right\} Q(z)$$

From equation (2.2) we get,

$$\frac{w_{n+1}^{(\alpha)}(v_j)}{v_j-\zeta} Q'(v_j) + \left\{\frac{[w_{n+1}^{(\alpha)}(v_j)]'}{v_j-\zeta} - \frac{w_{n+1}^{(\alpha)}(v_j)}{(v_j-\zeta)^2}\right\} Q(v_j) = 0$$

$$(v_j-\zeta) w_{n+1}^{(\alpha)}(v_j) Q'(v_j) + \left\{(v_j-\zeta)[w_{n+1}^{(\alpha)}(v_j)]' - w_{n+1}^{(\alpha)}(v_j)\right\} Q(v_j) = 0 \quad (2.4)$$

As v_j is a zero of $v_n^{(\alpha)}(z)$, we get

$(v_j + \alpha)^n = (1 + \alpha v_j)^n$ (2.5)

From equation (1.3) and (2.5) we get,

$w_{n+1}^{(\alpha)}(v_j) = (1+\alpha)(v_j+1)(v_j+\alpha)^n$(2.6)

$[w_{n+1}^{(\alpha)}(v_j)]' = (n+1)(1+\alpha)(v_j+\alpha)^n$ (2.7)

From equations (2.4), (2.6) and (2.7) we get,

$(v_j-\zeta)(v_j+1)Q'(v_j) + \{(n+1)(v_j-\zeta) - (v_j+1)\}Q(v_j) = 0,\ 1 \leq j \leq n$(2.8)

Now $Q(z) \in \pi_{n-1}$, the left hand side of equation belongs to π_{n-1} and has n zeros

Thus$Q(z)$ satisfies the following differential equation

$(z-\zeta)(z+1)Q'(z) + \{(n+1)(z-\zeta) - (z+1)\}Q(z) = C\, v_n^{(\alpha)}(z)$(2.9)

for some constant C. The integrating factor of differential equation (2.9) is given by

$$\varphi(z) = \frac{(z+1)^{n+1}}{(z-\zeta)}$$

The solution of differential equation (2.9) is given by

$$\varphi(z)Q(z) = C\int \frac{\varphi(t)\, v_n^{(\alpha)}(t)}{(t-\zeta)(t+1)}\, dt$$

$$\frac{(z+1)^{n+1}}{(z-\zeta)} Q(z) = C\int \frac{(t+1)^n}{(t-\zeta)^2}\, v_n^{(\alpha)}(t) dt$$

$$C\frac{(t+1)^n}{(t-\zeta)^2}\, v_n^{(\alpha)}(t)\Big|_{t=\zeta} = 0 \Rightarrow C = 0$$

$$Q(z) \equiv 0.$$

Theorem 2.2: Let $w_n^{(\alpha)}(z)$, $v_n^{(\alpha)}(z)$ be defined as in (1.1), (1.2) with $0 < \alpha < 1$ and $n \geq 2$, then the (0,1) Pál type Birkhoff interpolation problem for $\left\{ v_{2n+1}^{(\alpha)}(z), \frac{w_{2n}^{(\alpha)}(z)}{z-\zeta} \right\}$is regular.

Proof: Here we have total $2n + 1 + 2n - 1 = 4n$ interpolation points.

The problem is to find a polynomial $P(z) \in \pi_{4n-1}$ with

$P(v_i) = 0$ Where v_i is a zero of $v_{2n+1}^{(\alpha)}(z)$ (2.10)

$P'(w_j) = 0$ Where w_j is a zero of $\frac{w_{2n}^{(\alpha)}(z)}{z-\zeta}$(2.11)

Let us take $P(z) = v_{2n+1}^{(\alpha)}(z)Q(z), Q(z) \in \pi_{2n-2}$(2.12)

Thus $P(z) \in \pi_{4n-1}$.

The problem will be regular if $P(z) \equiv 0$.

Equation (2.12) gives,

$P'(z) = v_{2n+1}^{(\alpha)}(z)Q'(z) + [v_{2n+1}^{\alpha}(z)]'Q(z)$

By interpolatory condition (2.11), we have

$v_{2n+1}^{(\alpha)}(w_j)Q'(w_j) + [v_{2n+1}^{\alpha}(w_j)]'Q(w_j) = 0$ (2.13)

By equation (2.11) we get,

$(w_j + \alpha)^{2n} = -(1 + \alpha w_j)^{2n}$ (2.14)

Equations (1.2) and (2.14)

$v_{2n+1}^{(\alpha)}(w_j) = (1+\alpha)(1+w_j)(w_j+\alpha)^{2n}$(2.15)

$[v_{2n+1}^{\alpha}(w_j)]' = (2n+1)(1+\alpha)(w_j+\alpha)^{2n}$(2.16)

From equations (2.13), (2.15) and (2.16) we get,

$(w_j + 1)Q'(w_j) + (2n+1)Q(w_j) = 0, \qquad 1 \leq j \leq 2n-1$

Now $Q(z) \in \pi_{2n-2}$, the left hand side of equation belongs to π_{2n-2} and has $2n - 1$ zeros

Thus

$(z+1)Q'(z)+(2n+1)Q(z)=0$(2.17)

The integrating factor of differential equation (2.17) is given by

$$\varphi(z)=(z+1)^{2n+1}$$

The solution of differential equation (2.17) is given by

$Q(z)\varphi(z)=C$

$Q(z)=C(z+1)^{-2n-1}\in\pi_{2n-2}$

$Q(z)\equiv 0.$

Theorem 2.3: Let $w_n^{(\alpha)}(z)$, $v_n^{(\alpha)}(z)$ be defined as in (1.1), (1.2) with $0<\alpha<1$ and $n\geq 2$, then the (0,1) Pál type Birkhoff interpolation problem for $\left\{\frac{w_{2n+1}^{(\alpha)}(z)}{z-\zeta},\frac{v_{2n}^{(\alpha)}(z)}{z+1}\right\}$is regular.

Proof: Here we have total $2n+1-1+2n-1=4n-1$ interpolation points.

The problem is to find a polynomial $P(z)\in\pi_{4n-2}$ with

$P(w_i)=0$ Where w_i is a zero of $\frac{w_{2n+1}^{(\alpha)}(z)}{z-\zeta}$(2.18)

$P'(v_j)=0$ Where v_j is a zero of $\frac{v_{2n}^{(\alpha)}(z)}{z+1}$(2.19)

Let us take $P(z)=\frac{w_{2n+1}^{(\alpha)}(z)}{z-\zeta}Q(z)$, $Q(z)\in\pi_{2n-2}$(2.20)

Thus $P(z)\in\pi_{4n-2}$.

The problem will be regular if $P(z)\equiv 0$.

Equation (2.20) gives,

$$P'(z)=\frac{w_{2n+1}^{(\alpha)}(z)}{z-\zeta}Q'(z)+\left\{\frac{[w_{2n+1}^{(\alpha)}(z)]'}{z-\zeta}-\frac{w_{2n+1}^{(\alpha)}(z)}{(z-\zeta)^2}\right\}Q(z)$$

From equation (2.19) we get,

$$\frac{w_{2n+1}^{(\alpha)}(v_j)}{v_j-\zeta}Q'(v_j)+\left\{\frac{[w_{2n+1}^{(\alpha)}(v_j)]'}{v_j-\zeta}-\frac{w_{2n+1}^{(\alpha)}(v_j)}{(v_j-\zeta)^2}\right\}Q(v_j)=0$$

$(v_j-1)w_{2n+1}^{(\alpha)}(v_j)Q'(v_j)+\left\{(v_j-1)[w_{2n+1}^{(\alpha)}(v_j)]'-w_{2n+1}^{(\alpha)}(v_j)\right\}Q(v_j)=0$(2.21)

From equations (1.3) and (2.19) we get,

$w_{2n+1}^{(\alpha)}(v_j)=(1+\alpha)(v_j+1)(v_j+\alpha)^{2n}$ (2.22)

$[w_{2n+1}^{(\alpha)}(v_j)]'=(n+1)(1+\alpha)(v_j+\alpha)^{2n}$(2.23)

From equations (2.21), (2.22) and (2.23) we get,

$(v_j-\zeta)(v_j+1)Q'(v_j)+\{(2n+1)(v_j-\zeta)-(v_j+1)\}Q(v_j)=0,\qquad 1\leq j\leq 2n-1$

Now $Q(z)\in\pi_{2n-2}$, the left hand side of equation belongs to π_{2n-2} and has $2n-1$ zeros

Thus $Q(z)$ satisfies the following differential equation

$$(z-\zeta)(z+1)Q'(z)+\{(2n+1)(z-\zeta)-(z+1)\}Q(z)=C\frac{v_{2n}^{(\alpha)}(z)}{z+1} \qquad (2.24)$$

for some constant C. The integrating factor of differential equation (2.24) is given by

$$\varphi(z)=\frac{(z+1)^{2n+1}}{(z-\zeta)}$$

The solution of differential equation (2.24) is given by

$$\varphi(z)Q(z)=C\int\frac{\varphi(t)\,v_{2n}^{(\alpha)}(t)}{(t-\zeta)(t+1)}dt$$

$$\frac{(z+1)^{2n+1}}{(z-\zeta)}Q(z)=C\int\frac{(t+1)^{2n-1}}{(t-\zeta)^2}\,v_{2n}^{(\alpha)}(t)dt$$

$$C\frac{(t+1)^{2n-1}}{(t-\zeta)^2}\,v_{2n}^{(\alpha)}(t)\bigg|_{t=\zeta}=0\Rightarrow C=0$$

$$Q(z)\equiv 0.$$

Theorem 2.4: Let $w_n^{(\alpha)}(z)$, $v_n^{(\alpha)}(z)$ be defined as in (1.1), (1.2) with $0<\alpha<1$ and $n\geq 2$, then the (0,1) Pál type Birkhoff interpolation problem for $\left\{\frac{v_{n+1}^{(\alpha)}(z)}{z-1},\frac{w_n^{(\alpha)}(z)}{z-\zeta}\right\}$is regular.

Proof: Here we have total $2n-1$ interpolation points.

Let $P(z)=\frac{v_{n+1}^{(\alpha)}(z)}{z-1}Q(z)$ where $Q(z)\in\pi_{n-2}$(2.25)

Thus $P(z)\in\pi_{2n-2}$

With

$P(v_i)=0$ Where v_i is a zero of $\frac{v_{n+1}^{(\alpha)}(z)}{z-1}$(2.26)

$P'(w_j)=0$ Where w_j is a zero of $\frac{w_n^{(\alpha)}(z)}{z-\zeta}$(2.27)

From equations (2.25) and (2.27), we have

$$(w_j-1)v_{n+1}^{(\alpha)}(w_j)Q'(w_j)+\left\{(w_j-1)[v_{n+1}^{(\alpha)}(w_j)]'-v_{n+1}^{(\alpha)}(w_j)\right\}Q(w_j)=0 \quad (2.28)$$

By equations (2.14), (2.15), (2.16) and (2.28), we get

$$(w_j-1)(w_j+1)Q'(w_j)+\{(n+1)(w_j-1)-(w_j+1)\}Q(w_j)=0$$

Now $Q(z)\in\pi_{n-2}$, the left hand side of equation belongs to π_{n-2} and has $n-1$ zeros

Thus $Q(z)$ satisfies following differential equation

$$(z^2-1)Q'(z)+\{(n+1)(z-1)-(z+1)\}Q(z)=C\frac{w_n^{(\alpha)}(z)}{(z-\zeta)} \quad (2.29)$$

for some constant C. The integrating factor of differential equation (2.29) is given by

$$\varphi(z) = \frac{(z+1)^{n+1}}{(z-1)}$$

The solution of differential equation (2.29) is given by

$$\varphi(z)Q(z) = C\int \frac{\varphi(t)\, w_n^{(\alpha)}(t)}{(t-\zeta)(t^2-1)}dt$$

$$\frac{(z+1)^{n+1}}{(z-1)}Q(z) = C\int \frac{(t+1)^n}{(t-\zeta)(t-1)^2}\, w_n^{(\alpha)}(t)dt$$

$$C\frac{(t+1)^n}{(t-\zeta)(t-1)^2}\, w_n^{(\alpha)}(t)\Big|_{t=1,t=\zeta} = 0 \Rightarrow C = 0$$

$Q(z) \equiv 0.$

Theorem 2.5: Let $w_n^{(\alpha)}(z)$, $v_n^{(\alpha)}(z)$ be defined as in (1.1), (1.2) with $0 < \alpha < 1$ and $n \geq 2$, then the (0,1) Pál type Birkhoff interpolation problem for $\left\{\frac{v_{2n}^{(\alpha)}(z)}{z+1}, \frac{w_{2n-1}^{(\alpha)}(z)}{z-\zeta}\right\}$is regular.

Proof: Here we have total $4n-3$ interpolation points.

Let $P(z) = \frac{v_{2n}^{(\alpha)}(z)}{z+1}Q(z)$ where $Q(z) \in \pi_{2n-3}$

Thus $P(z) \in \pi_{4n-4}$

With

$P(v_i) = 0$ Where v_i is a zero of $\frac{v_{2n}^{(\alpha)}(z)}{z+1}$(2.30)

$P'(w_j) = 0$ Where w_j is a zero of $\frac{w_{2n-1}^{(\alpha)}(z)}{z-\zeta}$ (2.31)

From equation (2.31), we get

$(w_j+1)v_{2n}^{(\alpha)}(w_j)Q'(w_j) + \left\{(w_j+1)[v_{2n}^{(\alpha)}(w_j)]' - v_{2n}^{(\alpha)}(w_j)\right\}Q(w_j) = 0$(2.32)

By equations (2.14), (2.15), (2.16) and (2.32)

$(w_j+1)Q'(w_j) + 2nQ(w_j) = 0$

Now $Q(z) \in \pi_{2n-3}$, the left hand side of equation belongs to π_{2n-3} and has $2n-2$ zeros

Thus

$(z+1)Q'(z) + 2nQ(z) = 0$(2.33)

Integrating factor of differential equation (2.33) is given by

$$\varphi(z) = (z+1)^{2n}$$

The solution of differential equation (2.33) is given by

$Q(z)\varphi(z) = C$

$Q(z) = C(z+1)^{-2n} \in \pi_{2n-3}$

$Q(z) \equiv 0.$

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