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Indeterminate Forms and its Geometrical perspective

Hittu Gupta

Department of Mathematics , Khalsa College For Women, Civil lines, Ludhiana, Punjab, (India)

ABSTRACT

This paper deals with understanding of indeterminate forms from geometrical point of view using Menelaus' Theorem and gives a discussion on why $\frac{0}{0}$ *and* $\frac{\infty}{\infty}$ *are called indeterminate forms.*

Keywords :- Geometrical Perspective, Indeterminate forms, Menelaus' Theorem.

I. INTRODUCTION

Indeterminate forms are special cases of limits that occur when it is not possible to determine the limit value of an expression solely by recognizing the limiting behavior of its sub expressions. Possibly the most important indeterminate forms are the types $\frac{0}{0}$ and $\frac{\infty}{\infty}$ because to calculate the limit of the others, one should transform them to these types in order to apply L'Hopital's theorem.

II.INDETERMINATE FORMS

A mathematical expression can also be said to be indeterminate if it is not definitively or precisely determined. Certain forms of limits are said to be indeterminate when merely knowing the limiting behavior of individual parts of the expression is not sufficient to actually determine the overall limit. For example, a limit of the form 0/0, i.e., $\lim_{x\to 0} f(x)/g(x)$where $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, is indeterminate since the value of the overall limit actually depends on the limiting behavior of the combination of the two functions (e.g., $\lim_{x\to 0} x/x = 1$, while $\lim_{x\to 0} x^2/x = 0$).

There are seven indeterminate forms involving 0, 1, and ∞:

$$\frac{0}{0},\ 0\cdot\infty,\ \frac{\infty}{\infty},\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty$$

To understand the concept of indeterminate forms, firstly limit of a function at a point needs to be understood. Informally, a function *f* assigns an output *f*(*x*) to every input *x*. We say the function has a limit *L* at an input *p*: this means *f*(*x*) gets closer and closer to *L* as *x* moves closer and closer to *p*. More specifically, when *f* is applied to any input *sufficiently* close to *p*, the output value is forced *arbitrarily* close to *L*. On the other hand, if some inputs very close to *p* are taken to outputs that stay a fixed distance apart, we say the limit *does not exist*.

Suppose $f : \mathbf{R} \rightarrow \mathbf{R}$ is defined on the real line and $p, L \in \mathbf{R}$. It is said the limit of f, as x approaches p, is L and written

$$\lim_{x \to p} f(x) = L$$

if the following property holds:

For every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that for all real x, $0 < | x - p | < \delta$ implies $| f(x) - L | < \varepsilon$.

The value of the limit does not depend on the value of $f(p)$, nor even that p be in the domain of f.

A more general definition applies for functions defined on subsets of the real line. Let (a, b) be an open interval in $\mathbf{R}$, and p a point of (a, b). Let f be a real-valued function defined on all of (a, b) except possibly at p itself. It is then said that the limit of f, as x approaches p, is L if, for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that $0 < | x - p | < \delta$ and $x \in (a, b)$ implies $| f(x) - L | < \varepsilon$.

Here again the limit does not depend on $f(p)$ being well-defined.

Alternatively x may approach p from above (right) or below (left), in which case the limits may be written as

$$\lim_{x \to p^+} f(x) = L$$

or

$$\lim_{x \to p^-} f(x) = L$$

respectively. If these limits exist at p and are equal there, then this can be referred as *the* limit of $f(x)$ at p. If the one-sided limits exist at p, but are unequal, there is no limit at p (the limit at p does not exist). If either one-sided limit does not exist at p, the limit at p does not exist.

If f is a real-valued (or complex-valued) function, then taking the limit is compatible with the algebraic operations, *provided* the limits on the *right* sides of the equations below exist (the last identity only holds if the denominator is non-zero). This fact is often called the algebraic limit theorem.

$$\lim_{x \to p} (f(x)+ g(x)) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$$

$$\lim_{x \to p} (f(x) - g(x)) = \lim_{x \to p} f(x) - \lim_{x \to p} g(x)$$

$$\lim_{x \to p} (f(x). g(x)) = \lim_{x \to p} f(x) . \lim_{x \to p} g(x)$$

$$\lim_{x \to p} (f(x) / g(x)) = \lim_{x \to p} f(x) / \lim_{x \to p} g(x)$$

In each case above, when the limits on the right do not exist, or, in the last case, when the limits in both the numerator and the denominator are zero, nonetheless the limit on the left, called an ***indeterminate form,*** may still exist—this depends on the functions *f* and *g*. These rules are also valid for one-sided limits, for the case $p = \pm\infty$, and also for infinite limits using the rules

- $q + \infty = \infty$ for $q \neq -\infty$
- $q \times \infty = \infty$ if $q > 0$
- $q \times \infty = -\infty$ if $q < 0$
- $q / \infty = 0$ if $q \neq \pm \infty$

Note that there is *no* general rule for the case $q / 0$; it all depends on the way 0 is approached. Indeterminate forms---for instance, 0/0, $0\times\infty$, $\infty-\infty$, and ∞/∞— corresponding limits can often be determined with L'Hôpital's rule or the Squeeze theorem. Also we need to develop new techniques for dealing with problems where the usal methods to find limit fails.

III. L'HÔPITAL'S RULE

This rule uses derivatives to find limits of indeterminate forms 0/0 or $\pm\infty/\infty$, and only applies to such cases. Other indeterminate forms may be manipulated into this form. Given two functions $f(x)$ and $g(x)$, defined over an open interval *I* containing the desired limit point *c*, then if:

1. $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or

   $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \pm\infty$

2. f and g are differentiable over $\frac{I}{\{c\}}$ and

3. $g'(x) \neq 0$ for all $x \in \frac{I}{\{c\}}$ and

4. $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists

Then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

IV.NEED OF UNDERSTANDING INDETERMINATE FORMS USING GEOMETRICAL PERSPECTIVE

Since types of indeterminate forms and how to use L'Hopital's theorem to compute their value is covered in analysis courses at the university level. On several occasions during teaching, it has been seen that most of the students who were able to use L'Hopital's theorem to calculate the limits of indeterminate forms were unable to correctly grasp the meaning of indeterminate forms.

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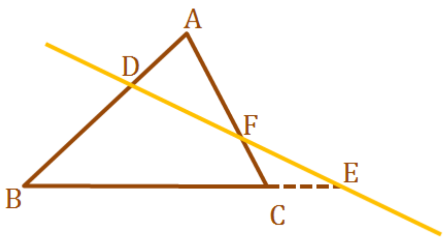
Namely, most students cannot interpret $\frac{\infty}{\infty}$ as the ratio of two quantities that increase without bound, or $\frac{0}{0}$ as ratio of two quantities both approaching zero. Instead, they tend to regard these symbols as a division operation on two numbers, such as $\frac{2}{5}$ or $\frac{0}{1}$. It always come to mind *that any number multiplied by zero is zero; why then do we regard 0.∞ as indeterminate? Should not the result be zero?*" or "*Division by zero is meaningless, so why are we searching for the limit value of a meaningless expression?*" or "*Is not the division of zero by a number zero? So why do we call it indeterminate*?"

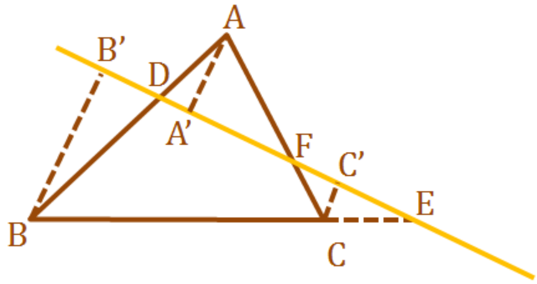
If it is asked to several people to answer the division of 7 by 0, surely only few could give a satisfactory explanation as to why this operation is undefined. A study on several mathematics students gave surprising results. Some were not able to answer the question, Some gave correct answer but they could not justify their answers; and some respond by saying "anything divided by zero is zero". In another study that focused on 35 in-service mathematics teachers, The findings revealed that only thirty-one percent of the teachers could give the correct answer and understood the underlying concept it is concluded that students cannot grasp the meaning of indeterminate forms, and the earlier mentioned studies offer some explanations for this situation. To make the basic understanding deep about concept, the indeterminate forms in geometrical context can serve the purpose using Menelaus' Theorem and students are made to experiment on geometrical constructs; and then students link their actions to the algebraic context.

Menelaus' theorem relates ratios obtained by a line cutting the sides of a triangle. The converse of the theorem is also true, and is extremely powerful in proving that three points are collinear. This theorem is widely applicable in various geometry problems.

Theorem 1. Menelaus' theorem states that if a line intersects Δ ABC or extended sides at points D, E and F the following statement holds: $\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} = 1$



Proof :- Construct lines AA',BB' and CC' that are perpendicular to the yellow line.



Now, since $\Delta AA'D \sim \Delta BB'D$

$$\frac{AD}{DB} = \frac{AA'}{BB'}$$

Since $\Delta AA'F \sim \Delta CC'F$

$$\frac{CF}{FA} = \frac{CC'}{AA'}$$

Since $\Delta BB'E \sim \Delta CC'E$

$$\frac{BE}{EC} = \frac{BB'}{CC'}$$

Hence $\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} = \frac{AA'}{BB'} \times \frac{BB'}{CC'} \times \frac{CC'}{AA'} = 1$

Note:

1. The equality still holds even when the yellow line does not intersect the triangle at all (that means the yellow line intersects at the extended parts of all sides of the triangle).
2. If the yellow line intersects one of the vertices of the triangle, then a 0 will appear in the denominator of the equation, which is undefined; which leads ***to indeterminate forms***, to solve this problem, the Menelaus' theorem could also be rewritten as .

$$AD \times BE \times CF = DB \times EC \times FA$$

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V.MENELAUS' THEOREM AS A WINDOW ON THE MEANINGS OF INDETERMINATE FORMS

Menelaus' theorem, named for Menelaus of Alexandria, is a theorem in plane geometry. The theorem states: Let ABC be a triangle, and F, E are two points on]AC[and]AB[respectively. If D is the intersection point of FE and CB, then . $\frac{|DB|}{|DC|}\frac{|CF|}{|FA|}\frac{|AE|}{|EB|} = 1$ ……….. (I)

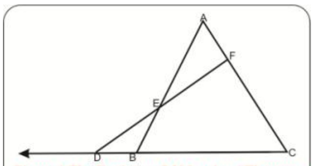

Figure 1.The Depiction Of Menelaus' Theorem

Although the theorem seems to be static on paper, it has a dynamic nature. This dynamism results from then arbitrariness of the positions of the points F and E. As a consequence of this arbitrariness, the theorem includes some of the indeterminate forms in its structure. First, imagine that, holding point D as fixed and keeping points D, E, and F collinear, point F gets closer and closer to point A. Then the ratio

$\frac{|CF|}{|FA|}$ increases without bound, and the ratio $\frac{|AE|}{|EB|}$ decreases continuously to zero. Therefore,

as F approaches A, the limit of $\frac{|CF|}{|FA|}\frac{|AE|}{|EB|}$ yields the indeterminate form $\infty.0$. Because the equality (I) holds irrespective of the positions of F and E, that limit is equal to the ratio $\frac{|DC|}{|DB|}$. In a similar manner, as point F approaches point A, the limit of $\frac{|CF|}{|FA|}$ yields the indeterminate form $\frac{\infty}{\infty}$. However, for the same reason, The limit is equal to the ratio $\frac{|DC|}{|DB|}$. Lastly, the indeterminate form $\frac{0}{0}$ comes from the ratio $\frac{|AE|}{|FA|}$ as point F approaches point A. As a result of possessing these indeterminate forms in its structure, the Menelaus' theorem is considered to be an appropriate context to concretely represent these abstract concepts in order to help students grasp the meaning.

Now discussion on various indeterminate forms is given :

5.1 why is $\frac{0}{0}$ indeterminate?

Theorem 2 :- *The number* $\frac{a}{b}$ *is defined, exists but not unique for* $a = 0$ *and* $b = 0$ *, that is,* $\frac{0}{0} = c$ *where* $c \in \mathbb{R}$

Proof. Let's take any arbitrary constants α and β such that $\alpha \neq \beta \neq 0$.

We know that $(0)^m = (0)^n$ then $(0)^m(\alpha) = (0)^n(\beta)$ if we divide the left and right hand side of this equation by $(0)^m$ and $(0)^n$ respectively, then we get

$$\left(\frac{0^m}{0^n}\right)\alpha = \left(\frac{0^m}{0^n}\right)\beta \Rightarrow \left(\frac{0}{0}\right)^m \alpha = \left(\frac{0}{0}\right)^n \beta$$

this implies that for every natural numbers *m* and *n*,

$$\left(\frac{0}{0}\right)^m(\alpha) - \left(\frac{0}{0}\right)^n(\beta) = 0 \Rightarrow \left(\frac{0}{0}\right)^m\{\alpha - \left(\frac{0}{0}\right)^{n-m}(\beta)\} = 0$$

$$\left(\frac{0}{0}\right)^m = 0 \text{ or } \alpha = \left(\frac{0}{0}\right)^{n-m}\beta$$

$$\Rightarrow \left(\frac{0}{0}\right) = 0 \text{ or } \left(\frac{0}{0}\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{n-m}}$$, which is a complex number

From this we must therefore conclude that $\left(\frac{0}{0}\right)$ is defined, exists but not unique so indeterminate.

5.2 why is $\frac{\pm\infty}{\pm\infty}$ indeterminate?

Theorem 3 :- *The number* $\frac{\frac{1}{a}}{\frac{1}{b}}$ *is defined, exists but not unique for* $a = 0$ *and* $b = 0$ *, that is,* $\frac{\pm\infty}{\pm\infty} = c$ *where* $c \in \mathbb{R}$

Proof. Let's take any arbitrary constants α and β such that $\alpha \neq \beta \neq 0$.

We know that $(\pm\infty)^m = (\pm\infty)^n$ then $(\pm\infty)^m(\alpha) = (\pm\infty)^n$ (β) if we divide the left and right hand side of this equation by $(\pm\infty)^m$ and $(\pm\infty)^n$ respectively, then we get

$$\left(\frac{\pm\infty^m}{\pm\infty^n}\right)\alpha = \left(\frac{\pm\infty^m}{\pm\infty^n}\right)\beta \Rightarrow \alpha = \left(\frac{\pm\infty}{\pm\infty}\right)^n \beta$$

this implies that for every natural numbers *m* and *n*, such that $m \neq n$

$$\left(\frac{\pm\infty}{\pm\infty}\right)^m(\alpha) - \left(\frac{\pm\infty}{\pm\infty}\right)^n(\beta) = 0 \Rightarrow \left(\frac{\pm\infty}{\pm\infty}\right)^m\{\alpha - \left(\frac{\pm\infty}{\pm\infty}\right)^{n-m}(\beta)\} = 0$$

$$\Rightarrow \left(\frac{\pm\infty}{\pm\infty}\right)^m = 0 \text{ or } \alpha = \left(\frac{\pm\infty}{\pm\infty}\right)^{n-m}\beta$$

$$\Rightarrow \left(\frac{\pm\infty}{\pm\infty}\right) = 0 \text{ or } \left(\frac{\pm\infty}{\pm\infty}\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{n-m}}$$, which is a complex number

From this we must therefore conclude that $\left(\frac{\pm\infty}{\pm\infty}\right)$ is defined, exists but not unique so indeterminate.

5.3 why is $(\pm\infty)$ - $(\pm\infty)$ indeterminate?

Theorem 4:- *The number* $\frac{1}{a} - \frac{1}{b}$ *is defined, exists but not unique for a* = 0 *and b* = 0 , *that is* , $(\pm\infty) - (\pm\infty)$ = *c where* $c \in \mathbb{R}$

Proof. Let's take any arbitrary constants α and β such that $\alpha \neq \beta \neq 0$.

We know that $((\pm\infty)-(\pm\infty))^m = ((\pm\infty)-(\pm\infty))^n$,

then $((\pm\infty)-(\pm\infty))^m(\alpha) = ((\pm\infty)-(\pm\infty))^n$ (β)

$$\Rightarrow ((\pm\infty)-(\pm\infty))^m(\alpha) - ((\pm\infty)-(\pm\infty))^n(\beta) = 0$$

this implies that for every natural numbers *m* and *n*, such that $m \neq n$

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$$\Rightarrow \left((\pm\infty)-(\pm\infty)\right)^m \{\ \alpha - \left((\pm\infty)-(\pm\infty)\right)^{m-n}\ (\beta)\} = 0$$

$$\Rightarrow \left((\pm\infty)-(\pm\infty)\right)^m = 0 \text{ or } \{\ \alpha - \left((\pm\infty)-(\pm\infty)\right)^{m-n}\ (\beta)\} = 0$$

$$\Rightarrow \left((\pm\infty)-(\pm\infty)\right) = 0 \text{ or } \left((\pm\infty)-(\pm\infty)\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{n-m}}$$, which is a complex number

From this we must therefore conclude that $\left((\pm\infty)-(\pm\infty)\right)$ is defined, exists but not unique so indeterminate.

Theorem 5 :- *The number* $\frac{a}{b}$ *doesn't exist for any positive real numbers* $a \neq 0$ *and* $b = 0$, *that is,* $\frac{a}{0} = \infty$.

Proof. We know that the laplace transform of 1 is equal to $\frac{1}{s}$ for $s \neq 0$, that is,

$$\int_0^\infty e^{-st}.1dt = \frac{1}{s}$$, for any s ,t $\in (0,\infty)$

But if $s = 0$, the laplace transform of 1 becomes doesn't exist, that is,

$$\int_0^\infty e^{-0t}dt = \frac{1}{0} \Rightarrow \lim_{k\to\infty} \int_0^k dt = \frac{1}{0} \Rightarrow \lim_{k\to\infty} (k) = \frac{1}{0} \Rightarrow \frac{1}{0} = +\infty$$

Therefore $\frac{1}{0}$ doesn't exist and if we multiply this equation both sides by any non zero positive real Number *a*, we get $\frac{a}{0} = +\infty$

Corollary :- $\frac{a}{0} = -\infty$, for a < 0

Proof: $a\left(\frac{1}{0}\right) = a\,(+\infty) \Rightarrow \frac{a}{0} = -\infty$, for any non zero negative real number a.

Theorem 6 :- *The number* $\frac{-a}{b}$ *doesn't exist for any positive real numbers* $a \neq 0$ *and* $b = 0$, *that is,* $\frac{-a}{0} = +\infty$.

Proof. We know that the laplace transform of 1 is equal to $\frac{1}{s}$ for $s \neq 0$, that is,

$$\int_0^{\infty} e^{-st}.1dt = \frac{1}{s} \text{ , for any } s,t \in (0,+\infty)$$

$\Rightarrow -\int_0^{\infty} e^{st}.1dt = \frac{1}{s}$, for any s ∈ (0,+∞) and t ∈ (- ∞ , 0) (integration by substitution)

$\Rightarrow \int_{-\infty}^{0} e^{st}.1dt = \frac{1}{s}$, for any s ∈ (0,+∞) and t ∈ (- ∞ , 0)

Let us replace s by –s, we get

$\int_{-\infty}^{0} e^{-st}.1dt = \frac{1}{-s} = \frac{-1}{s}$, for any s, t ∈ (- ∞ , 0)

But if $s = 0$, the laplace transform of 1 becomes doesn't exist, that is,

$$\int_{-\infty}^{0} e^{-0t}dt = \frac{-1}{0} \Rightarrow \lim_{k \to -\infty} (-k) = \frac{-1}{0} \Rightarrow \frac{-1}{0} = +\infty$$

Therefore $\frac{-1}{0}$ doesn't exist and if we multiply this equation both sides by any non zero positive real Number *a*, we get $\frac{-a}{0} = +\infty$

Corollary :- $\frac{-a}{0} = -\infty$, for a < 0

Proof: $a\left(\frac{-1}{0}\right) = a\,(+\infty) \Rightarrow \frac{-a}{0} = -\infty$, for any non zero negative real number a.

VI.CONCLUSION

Hence, aim of this paper is to introduce students to the geometrical representation of the basic calculus concept and this perspective can be beneficial while teaching the concept if students are asked to generate at least one example for each type of indeterminate forms as a geometrical construction.

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