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Unique fixed point theorems in dislocated quasi-metric space

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Abstract: *In this paper, the existence and uniqueness of a fixed point in a dislocated quasi-metric space are discussed for a single self-mapping using expanding and comparison function in the setting of dislocated quasi-metric space. These established results improve and modify some existing results in the literature.*

Keywords: *Dislocated quasi-metric space, contraction mapping, self-mappings, Cauchy sequence, fixed point.*

1. Introduction

Fixed point theory is one of the most crucial and dynamic research subjects of nonlinear analysis. In the area of fixed point theory, the first important and remarkable result was presented by Banach [3] for a contraction mapping in a complete metric space. Since then, a number of generalizations have been made by many researchers in their works. Matthews [9] introduced the concept of dislocated metric space with respect to metric. Abramsky and Jung [2] presented some facts about dislocated metric in the context of domain theory. Hitzler and Seda [6] generalized the celebrated Banach contraction principle in complete dislocated metric space. The notion of dislocated quasi-metric space was first time introduced by Zeyada *et al.* [14]. It is a generalization of the result due to Hitzler and Seda in dislocated metric space. Aage and Salunke [1] proved some results in dislocated and dislocated quasi-metric spaces. The purpose of this paper is to obtain some newly fixed point theorem for self-mapping in dislocated quasi-metric space using the concepts of expanding and comparison mappings. Examples are given in the support of our established results.

2. Preliminaries

Definition 2.1 [5] Let X be a non-empty set and let $d: X \times X \to [0, \infty)$ be a function satisfying following conditions:

(i) $d(x, y) = d(y, x) = 0, \Rightarrow x = y$.

(ii) $d(x, y) \leq d(x, z) = d(z, y), \forall x, y, z \in X$.

Then d is called a dislocated quasi-metric on X. If d satisfies $d(x, y) = d(y, x)$,
then it is called dislocated metric.

Definition 2.2 [5] A sequence $\{x_n\}$ is dislocated quasi-metric space (X, d) is called
Cauchy sequence if for a given $\varepsilon > 0$, there exists $n_0 \in N$, such that for all $m, n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$ i.e., $min\{d(x_m, x_n), d(x_n, x_m)\} < \varepsilon$.

Definition 2.3 [5] A sequence $\{x_n\}$ dislocated quasi-convergence to x if

$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$

In this case, x is called a dislocated quasi limit of $\{x_n\}$ and we write $x_n \to x$.

Lemma 2.1 [5] Dislocated quasi-limits in a dislocated quasi metric space are unique.

Definition 2.4 [5] A dislocated quasi-metric space (X, d) is called complete if every Cauchy sequence in it is a dislocated quasi-convergent.

Definition 2.5 [5] Let (X, d_1) and (X, d_2) be dislocated quasi-metric spaces and let $f: X \to Y$ be a function . Then f is continuous to $x_0 \in X$ if for each sequence $\{x_n\}$ which is d_1-quasi convergent to x_0, the sequence is $\{f(x_n)\}$ is d_2-quasi convergent to $f(x_0)$ in Y.

Definition 2.6 [5] let (X, d)be a dislocated quasi-metric space. $T: X \to X$ contraction if there exists $0 \le \lambda < 1$, such that $d(Tx, Ty) = d(x, y)$, for all $x, y \in X$.

3. Main Results

Theorem 3.1 Let (X, d) be a complete dislocated quasi metric space and let $T: X \to X$ be a mapping satisfying the following condition.

$$d(Tx,Ty) \ge \alpha \frac{[d(Tx,y)+d(x,y)][1+d(y,Ty)]}{1+d(x,y)} + \beta \frac{[d(Tx,y)+d(x,Tx)]d(y,Ty)}{d(x,y)} + \gamma \frac{d(Tx,y)+d(Tx,Ty)-d(Tx,y)^2 d(Tx,Ty)-d(Tx,y)d(Tx,Ty)^2}{1-d(Tx,y)d(Tx,Ty)} + \eta d(x,y) \qquad (3.1)$$

for all $x, y \in X;\ \alpha, \beta, \gamma, \eta \ge 0;\ \alpha+\beta+\gamma+\eta > 1$. Then T has a unique fixed point.

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows

Let $x_0 \in X, x_0 = Tx_1, x_1 = Tx_2, \ldots\ldots\ldots, x_n = Tx_{n+1}$

Consider $d(x_{n-1}, x_n) = d(Tx_n, Tx_{n+1})$

$$\alpha \frac{[d(Tx_n,x_{n+1})+d(x_n,x_{n+1})][1+d(x_{n+1},Tx_{n+1})]}{1+d(x_n,x_{n+1})} + \beta \frac{[d(Tx_n,x_{n+1})+d(x_n,Tx_n)]d(x_{n+1},Tx_{n+1})}{d(x_n,x_{n+1})} + \gamma \frac{d(Tx_n,x_{n+1})+d(Tx_n,Tx_{n+1})-d(Tx_n,x_{n+1})^2 d(Tx_n,Tx_{n+1})-d(Tx_n,x_{n+1})d(Tx_n,Tx_{n+1})^2}{1-d(Tx_n,x_{n+1})d(Tx_n,Tx_{n+1})} + \eta d(x_n,x_{n+1})$$

$$= \alpha \frac{[d(x_{n-1},x_{n+1})+d(x_n,x_{n+1})][1+d(x_{n+1},x_n)]}{1+d(x_n,x_{n+1})} + \beta \frac{[d(x_{n-1},x_{n+1})+d(x_n,x_{n-1})]d(x_{n+1},x_n)}{d(x_n,x_{n+1})} + \gamma \frac{d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)-d(x_{n-1},x_{n+1})^2 d(x_{n-1},x_n)-d(x_{n-1},x_{n+1})d(x_{n-1},x_n)^2}{1-d(x_{n-1},x_{n+1})d(x_{n-1},x_n)} + \eta d(x_n,x_{n+1})$$

$$= \alpha[d(x_{n-1},x_{n+1})+d(x_n,x_{n+1})] + \beta[d(x_{n-1},x_{n+1})+d(x_n,x_{n-1})] + \gamma[d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)] + \eta d(x_n,x_{n+1})$$

$$\ge \alpha d(x_{n-1},x_n) + \beta d(x_n,x_{n+1}) + \gamma d(x_n,x_{n+1}) + \eta d(x_n,x_{n+1})$$

$$= \alpha d(x_{n-1},x_n) + (\beta+\gamma+\eta) d(x_n,x_{n+1})$$

$$\Rightarrow (1-\alpha) d(x_{n-1},x_n) \ge (\beta+\gamma+\eta) d(x_n,x_{n+1})$$

$$d(x_n,x_{n+1}) \le \frac{(1-\alpha)}{(\beta+\gamma+\eta)} d(x_{n-1},x_n)$$

$$d(x_n,x_{n+1}) \le L d(x_{n-1},x_n)$$

Where $\frac{(1-\alpha)}{(\beta+\gamma+\eta)} = L, 0 \leq L < 1$

Similarly, we have

$d(x_{n-1}, x_n) \leq Ld(x_{n-2}, x_{n-1})$.

Continuing this process, we conclude that

$d(x_n, x_{n+1}) \leq L^n d(x_0, x_1)$

For $n > m$, using triangular inequality we have

$$d(x_n, x_m) \leq d(x_{n-1}, x_n) + d(x_{n-2}, x_{n-1}) + \dots\dots\dots\dots\dots + d(x_m, x_{m+1})$$

$$\leq [L^{n-1} + L^{n-2} + L^{n-3} + L^{n-4} + \dots\dots\dots\dots + L^m] d(x_0, x_1)$$

$$\leq \frac{L^m}{1-L} d(x_0, x_1)$$

For a natural number N_1 let $c < 0$ such that $\frac{L^m}{1-L} d(x_0, x_1) < c, \forall\, m \geq N_1$.

Thus $d(x_n, x_m) \leq \frac{L^m}{1-L} d(x_0, x_1) < c$ for $n > m$. Therefore $\{x_n\}$ is a Cauchy sequence in a complete dislocated quasi-metric space (X, d), $\exists\, z^* \in X$ such that $x_n \to z^*$ as $n \to \infty$. As T is continuous, so $T \lim_{n\to\infty} x_n = Tz^*$ implies $\lim_{n\to\infty} Tx_n = Tz^*$ implies $\lim_{n\to\infty} x_{n-1} = Tz^*$ implies $Tz^* = z^*$. Hence z^* is a fixed point T.

For uniqueness of fixed point z^*, let $z^{**}(z^* \neq z^{**})$ be another fixed point of T. We have to prove that $d(z^*, z^*) = d(z^{**}, z^{**}) = 0$

Consider $d(z^*, z^*) = d(Tz^*, Tz^*)$, then by (3.1) we have

$$d(z^*, z^*) \geq \alpha \frac{[d(Tz^*, z^*) + d(z^*, z^*)][1 + d(z^*, Tz^*)]}{1 + d(z^*, z^*)} + \beta \frac{d(Tz^*, z^*) + d(z^*, Tz^*)]d(z^*, Tz^*)}{d(z^*, z^*)}$$
$$+ \gamma \frac{d(Tz^*, z^*) + d(Tz^*, Tz^*) - d(Tz^*, z^*)^2 d(Tz^*, Tz^*) - d(Tz^*, z^*) d(Tz^*, Tz^*)^2}{1 - d(Tz^*, z^*) d(Tz^*, Tz^*)} + \eta d(z^*, z^*)$$

$$d(z^*, z^*) \geq (\alpha + \beta + \gamma + \eta) d(z^*, z^*)$$

Which is a contradiction, hence $d(z^*, z^*) = 0$. Similarly we, can show that $d(z^{**}, z^{**}) = 0$.

Now consider $d(z^*, z^{**}) = d(Tz^*, Tz^{**})$

$$\geq \alpha \frac{[d(Tz^*, z^{**}) + d(z^*, z^{**})][1 + d(z^{**}, Tz^{**})]}{1 + d(z^*, z^{**})} + \beta \frac{[d(Tz^*, z^{**}) + d(z^*, Tz^*)] d(z^{**}, Tz^{**})}{d(z^*, z^{**})}$$
$$+ \gamma \frac{d(Tz^*, z^{**}) + d(Tz^*, Tz^{**}) - (Tz^*, z^{**})^2 d(Tz^*, Tz^{**}) - d(Tz^*, z^{**}) d(Tz^*, Tz^{**})^2}{1 - d(Tz^*, z^{**}) d(Tz^*, Tz^{**})}$$
$$+ \eta d(z^*, z^{**})$$

$$\geq \alpha \frac{[d(z^*, z^{**}) + d(z^*, z^{**})][1 + d(z^{**}, z^{**})]}{1 + d(z^*, z^{**})} + \beta \frac{[d(z^*, z^{**}) + d(z^*, z^*)] d(z^{**}, z^{**})}{d(z^*, z^{**})}$$
$$+ \gamma \frac{d(z^*, z^{**}) + d(z^*, z^{**}) - d(z^*, z^{**})^2 d(z^*, z^{**}) - d(z^*, z^{**}) d(z^*, z^{**})^2}{1 - d(z^*, z^{**}) d(z^*, z^{**})} + \eta d(z^*, z^{**})$$

$$= \alpha [d(z^*, z^{**}) + d(z^*, z^{**})] + \gamma [d(z^*, z^{**}) + d(z^*, z^{**})] + \eta d(z^*, z^{**})$$

$$d(z^*, z^{**}) \geq 2(\alpha + \gamma) d(z^*, z^{**}) + \eta d(z^*, z^{**})$$

$$d(z^*, z^{**}) \geq [2(\alpha + \gamma) + \eta] d(z^*, z^{**})$$

Which is contradiction thus $d(z^*, z^{**}) = 0$

Similarly, we can show that $d(z^{**}, z^*) = 0$ which implies $z^* = z^{**}$ is the unique fixed point of T.

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Remark 3.1. If we put $\beta = \gamma = 0$ in theorem (3.1) we will get result of [11].

Example 3.1. Let $X = [0,1]$ with a complete dislocated-metric defined by

$d(x, y) = |x|$ for all $x, y \in X$,

and define the continuous self-mapping T by $Tx = \frac{x}{2}$ with $\alpha = \frac{1}{8}, \beta = \frac{1}{10}, \gamma = \frac{1}{12}$. Then T satisfies all the conditions of Theorem 3.1, and $x = 0$ is the unique fixed point of T in X.

Theorem 3.2. Let (X, d) be a complete dislocated quasi metric space and let $T: X \to X$ be a mapping satisfying the following condition.

$$d(Tx, Ty) \le q\, max \left\{ \begin{array}{c} \frac{d(x,y) + d(x,Ty)}{2}, \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(x,Tx)d(y,Ty)}{d(x,y)}, \\ \frac{d(x,y)\left[d(x,Tx) + \sqrt{d(Tx,Ty)d(y,Ty)}\right]^2}{[d(Tx,Ty) + d(x,y)]^2} \end{array} \right\} \quad (3.2)$$

Proof: For $x_0 \in X$ we define a sequence $\{x_n\}$ by $x_1 = Tx_0, x_2 = Tx_1, \dots \dots \dots \dots, x_{n+1} = Tx_n$.

Consider $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$$d(x_n, x_{n+1}) \le q\, max \left\{ \begin{array}{c} \frac{d(x_{n-1},x_n) + d(x_{n-1},Tx_n)}{2}, \frac{d(x_{n-1},Tx_{n-1}) + d(x_n,Tx_n)}{2}, \frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{d(x_{n-1},x_n)} \\ \frac{d(x_{n-1},x_n)\left[d(x_{n-1},Tx_{n-1}) + \sqrt{d(Tx_{n-1},Tx_n)d(x_n,Tx_n)}\right]^2}{[d(Tx_{n-1},Tx_n) + d(x_{n-1},x_n)]^2} \end{array} \right\}$$

$$= q\, max \left\{ \begin{array}{c} \frac{d(x_{n-1},x_n) + d(x_{n-1},x_{n+1})}{2}, \frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{2}, \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)} \\ \frac{d(x_{n-1},x_n)\left[d(x_{n-1},x_n) + \sqrt{d(x_n,x_{n+1})d(x_n,x_{n+1})}\right]^2}{[d(x_n,x_{n+1}) + d(x_{n-1},x_n)]^2} \end{array} \right\}$$

$$d(x_n, x_{n+1}) \le q\, max \left\{ \begin{array}{c} \frac{d(x_n,x_{n+1})}{2}, \frac{d(x_{n-1},x_{n+1})}{2}, \\ d(x_{n-1},x_n) \end{array} \right\}$$

$d(x_n, x_{n+1}) \le q\, d(x_{n-1}, x_n)$

In the similar fashion we can find

$d(x_n, x_{n+1}) \le q^n d(x_{n-1}, x_n)$

For $n > m$ we have

$d(x_n, x_m) \le d(x_{n-1}, x_n) + d(x_{n-2}, x_{n-1}) + \dots \dots \dots \dots \dots + d(x_m, x_{m+1})$

$\le [q^{n-1} + q^{n-2} + q^{n-3} + q^{n-4} + \dots \dots \dots \dots + q^m] d(x_0, x_1)$

$\le \dfrac{q^m}{1-q} d(x_0, x_1)$

For a natural number N_1 let $c < 0$ such that $\frac{q^m}{1-q} d(x_0, x_1) < c, \forall\, m \ge N_1$.

Thus $d(x_n, x_m) \le \frac{q^m}{1-q} d(x_0, x_1) < c$ for $n > m$. Therefore $\{x_n\}$ is a Cauchy sequence in a complete dislocated quasi-metric space (X, d), $\exists\, z^* \in X$ such that $x_n \to z^*$ as $n \to \infty$. i.e., $\lim_{n\to\infty} x_n \to z^*$ As T is continuous, so $T \lim_{n\to\infty} x_n = Tz^*$ implies $\lim_{n\to\infty} Tx_n = Tz^*$ implies $\lim_{n\to\infty} x_{n+1} = Tz^*$ implies $Tz^* = z^*$. Hence z^* is a fixed point T.

For uniqueness of fixed point z^*, let $z^{**}(z^* \neq z^{**})$ be another fixed point of T. We have to prove that $d(z^*,z^*) = d(z^{**},z^{**}) = 0$

Consider $d(z^*,z^*) = d(Tz^*,Tz^*)$, then by (3.2) we have

$$d(z^*,z^*) \leq q\, max \left\{ \begin{array}{c} \frac{d(z^*,z^*)+d(z^*,Tz^*)}{2}, \frac{d(z^*,Tz^*)+d(z^*,Tz^*)}{2}, \frac{d(z^*,Tz^*)d(z^*,Tz^*)}{d(z^*,z^*)}, \\ \frac{d(z^*,z^*)\left[d(z^*,Tz^*)+\sqrt{d(Tz^*,Tz^*)d(z^*,Tz^*)}\right]^2}{[d(Tz^*,Tz^*)+d(z^*,z^*)]^2} \end{array} \right\}$$

$$d(z^*,z^*) \leq q d(z^*,z^*)$$

Which is a contradiction, hence $d(z^*,z^*) = 0$. Similarly we can show that $d(z^{**},z^{**}) = 0$.

Now consider $d(z^*,z^{**}) = d(Tz^*,Tz^{**})$

$$\leq q\, max \left\{ \begin{array}{c} \frac{d(z^*,z^{**})+d(z^*,Tz^{**})}{2}, \frac{d(z^*,Tz^*)+d(z^{**},Tz^{**})}{2}, \frac{d(z^*,Tz^*)d(z^{**},Tz^{**})}{d(z^*,z^{**})} \\ \frac{d(z^*,z^{**})\left[d(z^*,Tz^*)+\sqrt{d(Tz^*,Tz^{**})d(z^{**},Tz^{**})}\right]^2}{[d(Tz^*,Tz^{**})+d(z^*,z^{**})]^2} \end{array} \right\}$$

$$d(z^*,z^{**}) \leq q\, max \left\{ \begin{array}{c} \frac{d(z^*,z^{**})+d(z^*,z^{**})}{2}, \frac{d(z^*,z^*)+d(z^{**},z^{**})}{2}, \frac{d(z^*,z^*)d(z^{**},z^{**})}{d(z^*,z^{**})} \\ \frac{d(z^*,z^{**})\left[d(z^*,z^*)+\sqrt{d(z^*,z^{**})d(z^{**},z^{**})}\right]^2}{[d(z^*,z^{**})+d(z^*,z^{**})]^2} \end{array} \right\}$$

$$d(z^*,z^{**}) \leq q d(z^*,z^{**})$$

This implies $z^* = z^{**}$ is a unique fixed point of T.

4. Conclusion

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