ABSTRACT
A vector space is a set \( V \) with two operations defined upon it; they are generally called as vector addition and scalar multiplication. Here the different conditions for the set to be a vector space are defined and verified. The importance of a vector space lies in the fact that many mathematical questions can be rephrased as a question about vector spaces. The geometric interpretation for elements of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) as points in the Euclidean plane and Euclidean space. For every given vector space there exists two subspaces which are called trivial subspaces. Here the properties of vector spaces are studied. The conditions for the existence of sums and direct sums are also verified. The finite and infinite dimensional vector spaces are studied using span. The linear dependence and independence of a set of vectors is verified. The terms Basis and Dimension of a vector space are discussed.

Key words: vector space, Euclidean space, Euclidean plane, subspace, span, Basis, dimension.

A vector space over \( F \) is a set \( V \) together with the two operations \( V \times V \rightarrow V \) and scalar multiplication \( F \times V \rightarrow V \) satisfying the following conditions.

1. **Commutative Property**: \( u+v = v+u; \ u,v \in V \)
2. **Associative Property**: \( u+(v+w)=(u+v)+w; \ u,v,w \in V \) and \( (ab)v = a(bv); \ a,b \in F \)
3. **Additive Identity**: There exists an element \( 0 \in V \) such that \( 0+v = 0, \ \forall v \in V \)
4. **Additive Inverse**: for every \( v \in V \), there exists and element \( w \in V \) such that \( v+w = 0 \)
5. **Multiplicative Identity**: \( I(v) = v; \ \forall v \in V \)
6. **Distributive Property**: \( a(u+v) = au + av \) and \( (a+b)u = au + bu; \ \forall a,b \in F, \ u,v \in V \)

A vector space over \( \mathbb{R} \) is called as a Real vector space. The elements \( v \in V \) are called vectors.

Example: consider \( F^n \), the set of all n-tuples. This is a vector space with addition and scalar multiplication.
The geometric interpretation for elements of $\mathbb{R}^2$ and $\mathbb{R}^3$ as points in the Euclidean plane and Euclidean space.

The following are the basic properties of vector spaces.

1. In every vector space additive identity is unique
2. For every $u \in V$ has a unique additive inverse
3. $0v = 0, \ \forall v \in V$
4. $0v = 0, \ \forall v \in V$
5. $(-1)v = -v$ for every $v \in V$

**Subspace:** Let $V$ be a vector space over $F$, and let $U \subset V$ be a subset of $V$. Then we say that $U$ is a subspace of $V$ if $U$ is a vector space over $F$ under the same operations that make $V$ into a vector space over $F$.

Let $U \subset V$ be a subset of a vector space $V$ over $F$. Then $U$ is a sub space of $V$ if and only if the following conditions are hold good:

1. Additive Identity: $0 \in U$
2. Closure under addition: $u, v \in U \Rightarrow u + v \in U$
3. Closed under the Scalar multiplication: $a \in F, u \in U \Rightarrow au \in U$

In every vector space $V$, the subsets $\{0\}, V$ are obviously forms a subspace. These are called as the trivial subspaces. Any other apart from these two are called as non trivial subspaces.

**Sum of Subspaces:** Let $U_1, U_2$ are the two sub spaces of $V$. Then the sum of the subspaces is denoted and defined as $U_1 + U_2 = \{u_1 + u_2 / u_1 \in U_1, u_2 \in U_2\}$.

**Direct Sum of Subspaces:** Suppose that every $u \in U$ can be written as $u = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$, then we write $U = U_1 \oplus U_2$ and we call it as the direct sum of $U_1$ & $U_2$.

Let $U_1$ & $U_2$ are the two sub spaces of $V$. Then $V = U_1 \oplus U_2$ if and only if the following two conditions are holds good.

1. $V = U_1 + U_2$
2. If $0 = u_1 + u_2; u_1 \in U_1 \& u_2 \in U_2$, then $u_1 = u_2 = 0$
Similarly the following conditions are also sufficient enough to define the direct sum of subspaces

Let $U_1$ & $U_2$ are the two subspaces of $V$. Then $V = U_1 \oplus U_2$ if and only if the following two conditions are holds good.

1. $V = U_1 + U_2$
2. $\{0\} = U_1 \cap U_2$

The linear span of $(v_1, v_2, v_3, \ldots, v_m)$ is defined as the set $\text{span}(v_1, v_2, v_3, \ldots, v_m) = \{a_1v_1 + a_2v_2 + \ldots + a_mv_m / a_1, a_2, \ldots, a_m \in F\}$

With the help of the above definition we can state the following lemma

Let $V$ is a vector space and $v_1, v_2, v_3, \ldots, v_m \in V$, then the following holds good

1. $v_j \in \text{span}(v_1, v_2, v_3, \ldots, v_m)$
2. $\text{span}(v_1, v_2, v_3, \ldots, v_m)$ is a subspace of $V$.
3. If $U \subset V$ is a subspace such that $v_1, v_2, v_3, \ldots, v_m \in U$, then $\text{span}(v_1, v_2, v_3, \ldots, v_m) \subset U$

Using the definitions of a span we can say that a vector space is a finite dimensional or infinite dimensional using the following criteria

If $\text{span}(v_1, v_2, v_3, \ldots, v_m) = V$, then we can say that $(v_1, v_2, v_3, \ldots, v_m)$ spans $V$ and we call $V$ as finite dimensional. otherwise it is called infinite dimensional.

For example
1. The vectors $e_1 = (1, 0, 0, 0, \ldots, 0), e_2 = (0, 1, 0, 0, \ldots, 0), \ldots, e_n = (0, 0, 0, 0, \ldots, 1)$ spans $F^n$. Hence $F^n$ is finite dimensional.

2. The vectors $V_1 = (1, 1, 0)$ and $V_2 = (1, -1, 0)$ span a subspace of $R^3$.

A list of vectors $(v_1, v_2, v_3, \ldots, v_m)$ is called linearly independent if the only solution for $a_1, \ldots, a_m \in F$ to the equation $a_1v_1 + a_2v_2 + \ldots + a_mv_m = 0$ is $a_1 = a_2 = \ldots = a_m = 0$.

In other words, the zero vector can only trivially be written as a linear combination of $(v_1, v_2, v_3, \ldots, v_m)$

A list of vectors $(v_1, v_2, v_3, \ldots, v_m)$ is called linearly dependent if it is not linearly independent. That is, $(v_1, v_2, v_3, \ldots, v_m)$ is linear dependent if there exist $a_1, \ldots, a_m \in F$, not all
zero, such that 
\[ a_1v_1 + a_2v_2 + \ldots + a_mv_m = 0 \]

**Basis:** A list of vectors \((v_1, v_2, v_3, \ldots, v_m)\) is a **basis** for the finite-dimensional vector space \(V\) if \((v_1, v_2, v_3, \ldots, v_m)\) is linearly independent and \(V = \text{span}(v_1, v_2, v_3, \ldots, v_m)\)

(Basis Reduction Theorem). If \(V = \text{span}(v_1, v_2, v_3, \ldots, v_m)\), then either \((v_1, v_2, v_3, \ldots, v_m)\) is a basis of \(V\) or some \(v_i\) can be removed to obtain a basis of \(V\)

And here we can observe that every finite dimensional vector space has a basis.

And we can prove some additional content related to the basis. One of the main concepts is Basis extension theorem.

It states that Every linearly independent list of vectors in a finite-dimensional vector space \(V\) can be extended to a basis of \(V\).

**Dimension:**
The length of any basis of a vector is said to be the dimension of that vector space. It is generally denoted by \(\text{dim}(V)\).

An important observation regarding the dimension of a vector space is that any two bases of a given vector space have the same dimension.

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