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Generalized Entropy of Kumaraswamy Distribution Based on Order Statistics

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ABSTRACT

Information theory is largely concentrated on uncertainty measurements or vulnerabilities of past and residual entropy functions. Many procedures such as survival function and the life testing problems have been put forward to compare the aging process of associated components and used to derive the remaining lifetime uncertainties. In this paper, entropy of order statistics, residual entropy and past residual entropy function based on Kumaraswamy (KS) distribution are proposed. Past residual entropy function is, however, subjected to upper bound computations and results are analyzed.

Keywords- Entropy, Order Statistics, Past Entropy, Residual Entropy

I INTRODUCTON

Characterization problems are largely affected by order statistics. These problems make immense use of analytical tools of mathematics and are located on the borderline that exists between statistics and probability. There are numerous outcomes on characterizations of probability of order statistics. For itemized review one may cite characterization results in view of cumulative residual entropy and non-additive entropy of order statistics [1] and [2].

The distribution theory of order statistics has been generally depicted in a few monographs composed by exceptional analysts and there are various papers committed to the hypothesis of order statistics and its applications and also asymptotic outcomes and implications in light of order statistics. One may mention order statistics properties of the Lomax appropriation [3] and Exponentiated Pareto distribution [4].

Order statistics have been utilized as a part of an extensive variety of issues, including firm statistical estimation, identification of anomalies, goodness of fit tests, examination of censored samples. Data properties of order statistics in view of Shannon entropy [5] and Kullback-Leibler [7] measure utilizing probability integral transformation have been examined by Ebrahimi et.al [6]. Two parametric generalized entropy, the Verma entropy [8] [9] and concentrated in context with order statistics. There are various principles of Shannon's entropy [5] which are available in the literature of information theory.

Definition1. A random variable y with arrange of values $[0,1]$ is said to have the kumaraswamy distribution, from now onwards abbreviated as KS, the probability density function (pdf) of KS distribution is

$$f(y;\alpha,\beta)=\alpha\,\beta\,y^{\alpha-1}\left(1-y^{\alpha}\right)^{\beta-1},\ 0<y<1 \tag{1}$$

Where α *and* β are the shape non negative shape parameters.

The cumulative distribution function (cdf) and the survival function of KS distribution is given by

$$F(y;\alpha,\beta)=1-\left(1-y^{\alpha}\right)^{\beta},\ 0<y<1 \tag{2}$$

and $$\bar{F}(y;\alpha,\beta)=\left(1-y^{\alpha}\right)^{\beta},\ 0<y<1 \tag{3}$$

The KS distribution is a continuous probability distribution with both its ends bounded. It is fundamentally the same as the beta distribution and can expect a strikingly vast variety of shapes and be utilized to show numerous arbitrary procedures and uncertainties. It has been used for Improved point estimation [10], Classical and Bayesian estimation [11], Prediction based on lognormal record values [12].

Definition2. An imperative idea in information theory introduced by Shannon $[8]$ is the notion of entropy. It assumes a fundamental part as a measure of intricacy and uncertainty in different fields such as material sciences, electronics and engineering to describe many chaotic systems. He proposed a measure of uncertainty for a non negative random variable Y, with a probability distribution having an absolutely continuous cumulative distribution function (cdf) $F(y)$ and probability distribution function (pdf) $f(y)$. Then the basic uncertainty measure is defined as,

$$H(Y)=-\int_{0}^{\infty} f(y)\log f(y)dy=-E[\log f(y)]. \tag{4}$$

Properties of Entropy Based on Order Statistics

Let $Y_1,Y_2,...,Y_n$ be a random sample from a distribution $F_Y(y)$ with density function $f_Y(y)>0$. The order statistics of this sample is defined by the arrangement of $Y_1,Y_2,...,Y_n$ from the smallest to the largest by $V_1<V_2<...<V_n$. The density of V_r, $r=1,2,...,n$ is

$$f_{V_r}(V)=\frac{n!}{(r-1)!(n-r)!}f_Y(y)[F_Y(y)]^{r-1}[1-F_Y(y)]^{n-r} \tag{5}$$

Now, let $U_1, U_2, ..., U_n$ be a random sample from $U(0,1)$ with the order statistics $Z_1 < Z_2 < ... < Z_n$. The density of $Z_r, r = 1, 2, ..., n$ is

$$f_{Z_r}(Z) = \frac{1}{B(r, n-r+1)} Z^{r-1}(1-Z)^{n-r}, 0 < Z < 1. \tag{6}$$

where $B(r, n-r+1) = \frac{\Gamma r \Gamma(n-r+1)}{\Gamma(n+1)} = \frac{(r-1)!(n-r)!}{n!}$

The entropy of the beta distribution is

$$H_n(Z_r) = -(r-1)[\psi(r) - \psi(n+1)] - (n-r)[\psi(n-r+1) - \psi(n+1)] + \log\beta(r, n-r+1)$$

Where $\psi(w) = \frac{d\log\Gamma w}{dw}$ *and* $\psi(n+1) = \psi(n) + \frac{1}{n}$

$$H(V_r) = H_n(Z_r) - E_{gr}\left[\log f_Y(F_Y^{-1}(Z_r))\right] \tag{7}$$

Using the substitution in equation (7) $Z_r = F_Y(y_r)$ and $Y_r = F_Y^{-1}(Z_r), r = 1, 2, ..., n$ is the probability integral transformations, the entropies of order statistics are obtained as:

For evaluating $H(V_r)$, we have $F_Y^{-1}(Z_r) = \left[1 - (1 - Z_r)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}$ and the expectation term in (7) is obtained as follows:

$$E_{g_r}\left[\log f_Y(F_Y^{-1}(Z_r))\right] = \left[\log(\alpha\beta) + \frac{(\beta-1)}{\beta}[\psi(n-r+1) - \psi(n+1)]\right.$$
$$\left. + \frac{(\alpha-1)}{\alpha}\left[\frac{n!}{(r-1)!}\sum_{i=0}^{n-r}\sum_{j=0}^{r+i-1}\binom{r+i-1}{j}\frac{1}{i!(n-r-i)!}(-1)^{i+j}[\psi(1) - \psi(\beta_j + \beta + 1)]\right]\right] \tag{8}$$

Using equation (8) in equation (7), we obtain

$$H(V_r) = \left[H_n(Z_r) - \log(\alpha\beta) - \frac{(\beta-1)}{\beta}[\psi(n-r+1) - \psi(n+1)]\right.$$
$$\left. - \frac{(\alpha-1)}{\alpha}\left(\frac{n!}{(r-1)!}\sum_{i=0}^{n-r}\sum_{j=0}^{r+i-1}\binom{r+i-1}{j}\frac{1}{i!(n-r-i)!}(-1)^{i+j}[\psi(1) - \psi(\beta_j + \beta + 1)]\right)\right] \tag{10}$$

For the sample minimum $r=1, H_n(Z_1)=1-\log n-\frac{1}{n}$.

$$H(V_1)=\left[1-\log n-\frac{1}{n}-\log(\alpha\beta)+\frac{(\beta-1)}{n\beta}\right.$$

$$\left.+\frac{(\alpha-1)}{\alpha}\left(\sum_{i=0}^{n-1}\sum_{j=0}^{i}\frac{(-1)^{i+j}n!}{(j)!(i-j)!(n-i-1)!}\left[-\psi(1)+\psi(\beta_j+\beta+1)\right]\right)\right] \quad (11)$$

For the sample maximum $r=n, H_n(Z_n)=1-\log n-\frac{1}{n}$.

$$H(V_n)=\left[1-\log n-\frac{1}{n}-\log(\alpha\beta)+\frac{(\beta-1)}{\beta}\left[-\psi(1)+\psi(n+1)\right]\right.$$

$$\left.+\frac{(\alpha-1)}{\alpha}\left(\sum_{j=0}^{n-1}\left(\frac{(-1)^j n!}{j!(n-j-i)!}\right)\left[-\psi(1)+\psi(\beta_j+\beta+1)\right]\right)\right] \quad (12)$$

Where $-\psi(1)=0.5772$ is the Euler's constant.

II RESIDUAL ENTROPY OF ORDER STATISTICS

In the event that a part is known to have survived to age *t*, at that point Shannon's Entropy isn't appropriate in order to measure the uncertainty of the remaining lifetime of the system. Therefore, Ebrahimi and Pellerey [13] characterized the residual entropy which measures the uncertainty in such cases. For a random lifetime of Y of system at time *t*, the residual entropy is defined as

$$H(Y;t)=-\int_t^{\infty} f_t(Y)\log f_t(Y)\,dy$$

where $f_t(Y)$ is the pdf of the variable $Y_t=\left(Y-t/Y>t\right)$ and is given by

$$f_t(Y)=\left\{\frac{f(y)}{\overline{F}(t)}, if\ Y>t\right\}$$

$$H(Y;t)=-\int_t^{\infty}\frac{f(y)}{\overline{F}(t)}\log\frac{f(y)}{\overline{F}(t)}\,dy, \quad t>0 \quad (13)$$

Where $\overline{F}(t)=1-\overline{F}(t)$ is the survival function of *Y*. The residual entropy of lifetime Y_t is given by (13). Analogous to expression (13), the residual entropy of order statistics $Y_{r,n}$ is given by

$$H(Y_{r,n};t)=-\int_t^{\infty}\frac{f_{r,n}(y)}{\overline{F}_{r,n}(t)}\log\frac{f_{r,n}(y)}{\overline{F}_{r,n}(t)}dy,\quad t>0 \tag{14}$$

It is clear that the residual entropy of first order statistics is obtained by substituting $r=1$, then using probability integral transformation $Z=F_Y(y)$ in equation (14) and using equations (3) and (6), we have

$$H(Y_{1,n};t)=\left(\frac{n-1}{n}\right)-\log n+\log\left(\overline{F}(t)\right)-\left(\frac{n}{\overline{F}^{n}(t)}\right)\int_{F(t)}^{1}(1-z)^{n-1}\log f\left(F^{-1}(z)\right)dz \tag{15}$$

where

$$\left(\frac{n}{\overline{F}^{n}(t)}\right)\int_{F(t)}^{1}(1-z)^{n-1}\log f\left(F^{-1}(z)\right)dz=\left[\log(\alpha\beta)-n\frac{(\alpha-1)}{\alpha}\sum_{s=1}^{\infty}\frac{\left(\overline{F}(t)\right)^{\frac{s}{\beta}}}{s\left(n+\frac{s}{\beta}\right)}\right.$$

$$\left.+\frac{n(\beta-1)}{\beta\left(\overline{F}^{n}(t)\right)}\sum_{l=0}^{n-1}\sum_{s=2}^{\infty}\binom{n-1}{l}(-1)^{l}\left(\frac{F^{l+s}(t)-1}{(s-1)(l+s)}\right)\right] \tag{16}$$

Using equation (16) in equation (15), we get

$$H(Y_{1,n};t)=\left[\left(\frac{n-1}{n}\right)-\log n+\beta\log\left(1-t^{\alpha}\right)-\log(\alpha\beta)-n\left(\frac{\alpha-1}{\alpha}\right)\sum_{s=1}^{\infty}\frac{\left(1-t^{\alpha}\right)^{s}}{s\left(n+\frac{s}{\beta}\right)}\right.$$

$$\left.-\frac{(\beta-1)}{\beta}\frac{n}{\left(1-t^{\alpha}\right)^{\beta}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^{l}\left[\frac{\left(1-\left(1-t^{\alpha}\right)^{\beta}\right)^{l+s}-1}{(s-1)(l+s)}\right]\right]$$

III PAST ENTROPY

It is reasonable to assume that in numerous circumstances uncertainty isn't really identified with future yet can likewise allude to past For instance, if *Y* denotes the lifetime of a system or of living organism, Y at time t, a system is observed only at certain preassigned times, it is observed to be down, then the uncertainty of the system life relies i.e., on which moment in (0,t), it has failed. In view of the thought, Dicrescenzo and Longobardi [14] have contemplated the past entropy over $(0,t)$ and is characterized as

$$H^{o}(Y;t)=-\int_{0}^{t}\frac{f(y)}{F(t)}\log\frac{f(x)}{F(t)}dy \tag{17}$$

where $F(t)$ is the cumulative distribution function. Analogous to expression (17) the past residual entropy of the r^{th} order statistics is defined as

$$H^{o}(Y_{r,n};t)=-\int_{0}^{t}\frac{f_{r,n}(y)}{\bar{F}_{r,n}(t)}\log\frac{f_{r,n}(y)}{\bar{F}_{r,n}(t)}dy, \tag{18}$$

It is clear that the past residual entropy of n^{th} order statistics is obtained by substituting $r=n$ and using probability integral transformation $z=F_{Y}(y)$ in equation (18) and using equations (3) an, we have

$$H^{o}(Y_{n,n},t)=\left(\frac{n-1}{n}\right)-\log n+\log(F(t))-\frac{n}{F^{n}(t)}\int_{0}^{F(t)}z^{n-1}\log\{f(F^{-1}(z))\}dz$$

(19)

$$\frac{n}{F^{n}(t)}\int_{0}^{F(t)}z^{n-1}\log\{f(F^{-1}(z))\}dz=\left[\log\alpha\beta+\frac{n}{F^{n}(t)}\left(\frac{\alpha-1}{\alpha}\right)\sum_{l=0}^{n-1}\sum_{s=1}^{\infty}\binom{n-1}{l}(-1)^{l}\left[\frac{\bar{F}(t)^{(l+s)+\frac{s}{\beta}}-1}{s\left((l+s)+\frac{s}{\beta}\right)}\right]\right.$$

$$\left.-\left(\frac{n(\beta-1)}{\beta}\right)\sum_{s=1}^{\infty}\frac{F(t)^{s}}{s(n+s)}\right] \tag{20}$$

Therefore, using equation (20) in equation (19), we get

$$H^{o}(Y_{n,n},t)=\left[\left(\frac{n-1}{n}\right)-\log n+\log\left(1-(1-t^{\alpha})^{\beta}\right)-\log(\alpha\beta)+\frac{n(\beta-1)}{\beta}\sum_{s=1}^{\infty}\frac{\left(1-(1-t^{\alpha})^{\beta}\right)^{s}}{s(n+s)}\right.$$

$$\left.-n\left(\frac{\alpha-1}{\alpha}\right)\sum_{l=0}^{n-1}\sum_{s=1}^{\infty}\binom{n-1}{l}\frac{(-1)^{l}}{1-(1-t^{\alpha})^{\beta}}\left[\frac{(1-t^{\alpha})^{\beta(l+s)+s}-1}{s\left((l+s)+\frac{s}{\beta}\right)}\right]\right] \tag{21}$$

An Upper Bound to the Past Entropy of Order Statistics

We present the upper bound for the past entropy of order statistics under the condition that $f_{r,n}\le 1$

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$$H^0\left(y_{r,n};t\right)=-\int_0^t \frac{f_{r,n}(y)}{F_{r,n}(t)}\log\frac{f_{r,n}(y)}{F_{r,n}(t)}dy$$

$$=\log\left(F_{r,n}(t)\right)-\frac{1}{F_{r,n}(t)}\int_0^t f_{r,n}(y)\log(y)dy$$

For $t>0$, $\log\ F_{r,n}(t)\leq 0$. We obtain

$$H^0\left(y_{r,n};t\right)\leq\frac{H\left(y_{r,n}\right)}{F_{r,n}(t)} \tag{22}$$

Substituting $r=1$ in equation (22) and using equations (2) and (6) and probability integral transformation, we obtain

$$H^0\left(y_{1,n};t\right)\leq\frac{-1}{F_{1,n}(t)}\Bigg[\log(n)+n\log(\alpha\beta)-n\left(\frac{\alpha-1}{\alpha}\right)\sum_{s=1}^{\infty}\frac{\beta}{s(n\beta+s)}$$

$$+\frac{(\beta-1)}{\beta}\sum_{l=0}^{n-1}\sum_{s=2}^{\infty}\binom{n-1}{l}\left(-1^{l+1}\frac{1}{(s+l)(s-1)}\right)\Bigg]$$

$$H^0\left(y_{1,n};t\right)\leq\frac{1}{\left(1-\left(1-t^{\alpha}\right)^{n\beta}\right)}\Bigg[n\left(\frac{\alpha-1}{\alpha}\right)\sum_{s=1}^{\infty}\frac{\beta}{s(n\beta+s)}-\log(n)-n\log(\alpha\beta)$$

$$-\frac{(\beta-1)}{\beta}\sum_{l=0}^{n-1}\sum_{s=2}^{\infty}\binom{n-1}{l}\left((-1)^{l+1}\frac{1}{(s+l)(s-1)}\right)\Bigg]$$

Substituting $r=n$ in equation (22) and using equations (2) and (6) and probability integral transformation, we obtain

$$H^0\left(y_{n,n};t\right)\leq\frac{-1}{F_{n,n}(t)}\Bigg[\log(n)+\log(\alpha\beta)+n(n-1)\sum_{l=0}^{n-1}\sum_{s=2}^{\infty}\binom{n-1}{l}(-1)^{l+1}\frac{1}{(s-1)(s+l)}$$

$$+n\frac{(\alpha-1)}{\alpha}\sum_{l=0}^{n-1}\sum_{s=1}^{\infty}\binom{n-1}{l}(-1)^{l}\frac{1}{s\left((l+s)+\frac{s}{\beta}\right)}-n\frac{(\beta-1)}{\beta}\sum_{s=1}^{\infty}\frac{1}{s(n+s)}\Bigg]$$

$$H^0\left(y_{n,n};t\right)\leq\frac{1}{\left(1-\left(1-t^{\alpha}\right)^{\beta}\right)^{n}}\left[n\frac{(\beta-1)}{\beta}\sum_{s=1}^{\infty}\frac{1}{s(n+s)}-n\frac{(\alpha-1)}{\alpha}\sum_{l=0}^{n-1}\sum_{s=1}^{\infty}\binom{n-1}{l}(-1)^{l}\frac{1}{s\left((l+s)+\frac{s}{\beta}\right)}\right.$$

$$\left.-\log(n)-\log(\alpha\beta)-n(n-1)\sum_{l=0}^{n-1}\sum_{s=2}^{\infty}\binom{n-1}{l}(-1)^{l+1}\frac{1}{(s-1)(s+l)}\right]$$

IV CONCLUSION

we have presented and considered the entropy of order statistics based on kumaraswamy distribution. We have proposed some preliminary results of residual entropy, past residual entropy functions of (KS) distributions. We also analyze the results of upper bound of the past residual entropy functions. The hypothetical outcomes got in this paper can be utilized to go further and investigate the applications in different orders where the uncertainty exists.

REFERENCES

[1] Thapliyal R.,Taneja H.C.(2015). Characterization results based on non-additive entropy of order statistics. *Elsevier Physica*, (417), 1-366.

[2] Osmani Kamri (2016). On Dynamic Cumulative Residual Entropy of Order Statistics, *Statistics Applications & probability*, (3), 515-519.

[3] Javid D., Bander Al-Zahrani. (2016). On Some Order Statistics Properties othe Mukherjee-Islam distribution, *J. Math. Comput. Sci. 6*, (4), 555-567.

[4] Behboodian J., Tahmasebi. (2008). Some Properties of Entropy for the Exponential Pareto Distribution based on Order Statistics. *Mathematical Extension*, (3), 43-53.

[5] Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal*, (27), 379-423 and 623-656.

[6] Ebrahimi, N. et.,al (2004). Information properties of order statistics and spacing. *IEEE Trans. Information Theory*, (50), 177-183.

[7] Kullback.S. (1959). Information Theory and Statistics. *Wiley, New York.*

[8] Thapliyal R.,Taneja H.C. (2012) Generalized entropy of order statistics. *Applied Mathematics*,3,1977-1982.

[9] Verma R.S. (1966) Generalization of Renyi's Entropy of Order α. *Journal of Mathematical Sciences*, Vol. 1, pp. 34-48.

[10] Artur J. Lemonte,(2011), Improved point estimation for the Kumaraswamy distribution, *Statistical Computation and Simulation*, (12), 1971-1982

[11] Sukhdev S., Yogesh M., (2017). Bayesian estimation and prediction based on lognormal record values. *Journal of Applied probability*, (5), 916-940.

[12] Kızılaslan F., & Nadar M., (2016). Estimation and prediction of the Kumaraswamy distribution based on record values and inter record times. *Statistical Computation and Simulation*, (12),1563-5163

[13] Ebrahimi, N. and Pellerey, F., (1995), New partial ordering of survival functions based on the notion of uncertainty. *Journal of Applied probability*, (32), 202-211

[14] Crescenzo, A.D. and Longobardi, M., (2004), A measure of discrimination between past life time distributions, *Stat.Prob.Lett*. (67), 173-182