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On the Conjugacy Classes, Centers and Representation of the Groups Sn and Dn and the subgroups of the Dihedral group Dn.

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ABSTRACT

This research paper determines a formula for the number of subgroups of the dihedral groups D(n). I created the operation tables and lattice of subgroups for D(3) through D(8). After creating the lattice of subgroups I determined the elements of D(n) that generate each subgroup of D(n). This led to the formula $S_n = \tau(n) + \sigma(n)$, where S_n represents the number of subgroups of D(n), $\tau(n)$ represents the number of positive divisors of n, and $\sigma(n)$ represents the sum of the positive divisors of n. It has always been a difficult task in determining the behaviors of reflections and rotational symmetries in these symmetry groups and how much information can be obtained from there symmetries. We therefore study the nature and properties of these symmetry elements including the conjugacy class size in both Sn and Dn. It was found that the conjugacy classes of Sn are determined by their cycle type while that of Dn is a special case, where the relation "Conjugacy" is an equivalence relation. The representations of the conjugacy class size of Dn reveals that the order of the centers of Dn are 1 (for nodd) and 2 (for n-even), and consequently, leading to two different class equations of Dn.This paper will conclude with a description of how D(n) can be utilized in a secondary classroom. Since abstract algebra is not a topic primarily focused upon in secondary education, this section will contain a lesson in which students will be asked to determine the symmetries/permutations of various figures. During this lesson students will be introduced to the definition of line of symmetry, rotational symmetry, and composition. Students will be given various figures and asked to list all of the symmetries for each. Next, students will create a system to list the permutations of certain regular polygons. The target learning goal of this lesson is for students to identify the combination of permutations that yield the identity of each figure. The extension of this lesson will be for students to relate the permutations of a two-dimensional figure to a three-dimensional figure.

Keywords: {Dihedral Group D(n), $\tau(n)$, $\sigma(n)$, S_n, Permutations, Subgroups}

I.INTRODUCTION

Given any set X and a collection G of all bijections of X onto itself (also known as permutation) that is closed under compositions and inverses, G is a group acting on X. If X consists of n elements and G consists of all permutations, then G is the symmetric group Sn, generally referred by Lang, (2005) as subgroup of the symmetric group of X. Permutation groups and matrix groups are special cases of transformation groups; group that act on a certain space X preserving its inherent structure. In the case of permutation groups, X is a set. An early construction due to Cayley, exhibited any finite group as a permutation group acting on itself (X=G) (Nummela, 1980). The concept of transformation group is closely related to the concept of symmetric group. Transformation groups frequently consist of all transformations that preserve a certain structure (Robinson, 1996). In group theory, a dihedral group is the group of symmetries of a regular polygon, including both rotations and reflections (Dummit, 2004). Dihedral groups are among the simplest examples of finite Int.

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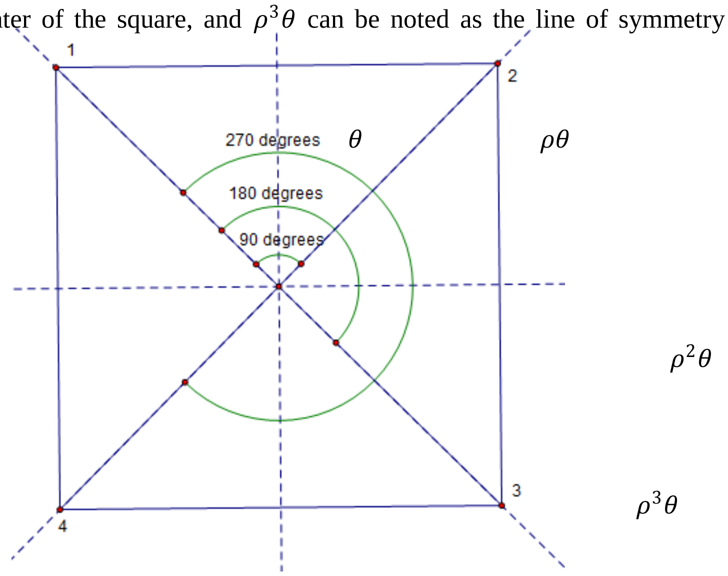
This project will make use of the definition that all of the permutations for each of the dihedral groups D(n) preserve the cyclic order of the vertices of each regular n-gon. This demonstrates the relationship between the abstract concept of D(n) and the rigid motions of a regular n-gon. From this, I will denote the rotation symmetry for each of the regular n-gons of $\frac{2\pi}{n}$ (clockwise) as ρ, the reflection symmetry as θ, and the original n-gon as ε. For example, in the following diagram the identity of the square (ε) is the square in which each vertex and side is matched. Since a square has 4 sides, the rotation ρ is equal to $\frac{2\pi}{4}$ which is a 90^0 clockwise rotation, $\rho^2$is equivalent to a 180^0 clockwise rotation, ρ^3 is equivalent to a 270^0 clockwise rotation, and ρ^4 is equivalent to a 360^0 clockwise rotation which is equivalent to the identity ε. θ can be noted as the reflection about the vertical line which passes through the center of the square (as seen below), $\rho\theta$ can be noted as the line of symmetry which passes through the vertices 2 and 4, $\rho^2\theta$ can be noted as the horizontal line of symmetry which passes through the center of the square, and $\rho^3\theta$ can be noted as the line of symmetry which passes through the vertices 1 and 3.



The Dihedral Group D(n)

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I will begin this section by describing what each dihedral group D(n) represents. I will define the notation used to create each group D(n), the operation tables, and the lattice of the subgroups for each n. From this, the formula that outputs the number of subgroups of the dihedral group D(n) will be conjectured.

As described in the background information, the two types of symmetries that regular polygons have are rotational symmetry and line symmetry. Each of the rotational symmetries will be labeled as the powers of ρ, each line of symmetry will be labeled as the powers of ρ times θ, and ε will represent the original regular polygon such that the vertices are in their original circular order. Each of the dihedral groups will be represented in this paper using this notation.

To determine the number of subgroups of D(n) and the process to derive the formula I will identify representations for each dihedral group D(n). This includes a) a picture of the regular polygons, b) the elements contained in the group D(n), c) the operation table, and d) the lattice of the subgroups for each D(n). The operation tables define all of the symmetry operations. The operation tables are used to identify the closure of a set; subsequently, identifying all of the subgroups. The lattices of the subgroups begin with the entire group D(n) and will branch to each subgroup that is a subset (or contained) in the group connected above until it reaches the identity ε. The representation for a clockwise rotation of a regular triangle, ρ, will be represented as $\binom{123}{231}$ or 1→2, 2→3 and 3→1

Group Structure of Dn

The composition of two symmetries of a regular polygon is again symmetry, as in the case of geometric object. It is the result of this operation that gives the symmetries of a regular polygon the algebraic structure of a finite group (Samaila, 2010). The composition operation is not commutative, and in general, the group Dn has the following elements: Dn = {r0 = e, r1, r2, …, rn-1, f0, f2, …, fn-1} with the following properties: ri rj = r(i+j)modn; ri fj = f(i-j)modn; fj ri = f(j-i)modn, fi fj = r(i-j)modn. The 2n elements of Dn can be written as e, r, r 2 , …, r n-1 , f, rf, r 2 f, …, r n-1f. The first n elements are the elements of the rotations and the remaining n elements are axes reflections (all have order 2). Obviously, the product of two rotations or two reflections is a rotation, while the product of a rotation and a reflection is a reflection. From the information provided so far on Dn, it is therefore convenient to write Dn as Dn = 〈 r, f | r n = e = f 2 , f rf = r -1 , rfr = f 〉 1 The group with representation as in equation 1 above or as Dn = 〈 x,y|x 2 = y 2 = (xy) = e〉 2 From the second presentation, it follows that Dn belongs to the class of Coxeter groups.

The Collection of the Number of Subgroups of Dihedral Group D(n)

n	Number of Subgroups of D(n)
3	6
4	10
5	8
6	16

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7	10
8	19

Now that D(3) through D(8), along with their subgroups have been described, I will now investigate the mathematics behind creating a formula that outputs the number of subgroups for D(n). The formula was contrived through trial and error while I was trying to generate the list of subgroups of D(n). I quickly noted that D(n) will always contain the subgroup D(n), the subgroup ε, and the subgroups generated by ρ^{a_i} where a_i are the positive divisors of n. If k is relatively prime to n then no additional subgroups can be generated by ρ^k. Modular arithmetic demonstrates that a relatively prime number will generate every number contained in the set created by mod(n); therefore, each subgroup corresponds to a factor of n.

I will investigate the subgroups for D(4) and D(8). It is noted for D(4) that the factors of 4 are 1,2, and 4. The subgroups of D(4) are as follows; $\{D_4\},\{\rho,\rho^2,\rho^3,\varepsilon\},\{\rho^2,\varepsilon\},\{\varepsilon\},\{\rho^2,\theta,\rho^2\theta,\varepsilon\},\{\rho^2,\rho\theta,\rho^3\theta,\varepsilon\},\{\theta,\varepsilon\},\{\rho\theta,\varepsilon\},\{\rho^2\theta,\varepsilon\},\{\rho^3\theta,\varepsilon\}$. I will break these subgroups into two groups: a) subgroups that only contain rotations and b) subgroups that contain reflections.

a) Looking at the three subgroups which contain rotations of the square; ρ will generate the subgroup only containing rotations generated by a 90^0 clockwise rotation, ρ^2 will generate the subgroup of rotations generated by a 180^0 clockwise rotation, and ρ^4 (or ε) will generate the last subgroup that is generated by a 360^0 clockwise rotation. Thus I can conjecture that the number of subgroups of D(4) that only contain rotations is equivalent to the number of factors of 4.

b) I will now investigate the subgroups that contain rotations and reflections. The subgroup generated by ρ and θ will produce the the entire group D(n). The subgroup generated by ρ^2 and θ will produce $\{\rho^2,\theta,\rho^2\theta,\varepsilon\}$. The subgroup generated by $\rho^2\, and\, \rho\theta$ will produce$\{\rho^2,\rho\theta,\rho^3\theta,\varepsilon\}$. The subgroups generated by ρ^4 or ε and each individual reflection are $\{\theta,\varepsilon\},\{\rho\theta,\varepsilon\},\{\rho^2\theta,\varepsilon\}, and\ \{\rho^3\theta,\varepsilon\}$. All things considered, I am able to conjecture that the number of subgroups of D(4) is equivalent to 3+1+2+4. This is equivalent to the number of factors of 4 plus each factor of 4.

Now that I have investigated the number of subgroups for D(4), D(8) will be explored where there are a total of 19 subgroups. Throughout this section I will examine two categories of subgroups: a) subgroups that only contain rotations and b) subgroups that contain reflections. Identifying how each subgroup of D(8) is generated will reveal the formula that outputs the number of subgroups of D(8).

a) In D(8) the subgroups that only contain rotations are $\{\rho,\rho^2,\rho^3,\rho^4,\rho^5,\rho^6,\rho^7,\varepsilon\},\{\rho^2,\rho^4,\rho^6,\varepsilon\},\{\rho^4,\varepsilon\}\ and\ \{\varepsilon\}$. The subgroup $\{\rho,\rho^2,\rho^3,\rho^4,\rho^5,\rho^6,\rho^7,\varepsilon\}$ represents the subgroup of rotations generated by ρ, $\{\rho^2,\rho^4,\rho^6,\varepsilon\}$ is the subgroup of rotations that is generated by ρ^2, $\{\rho^4,\varepsilon\}$ is the subgroup of rotations that is generated by ρ^4, and $\{\varepsilon\}$ is the subgroup that is generated by ρ^8 or the identity. The importance of this section is to realize that D(8) has four subgroups that only contain rotations. Notice that eight has four factors of 1, 2, 4 and 8.

b) The subgroups that contain both rotations and reflections are $\{D_8\}, \{\rho^2, \rho^4, \rho^6, \theta, \rho^2\theta, \rho^4\theta, \rho^6\theta, \varepsilon\}, \{\rho^2, \rho^4, \rho^6, \rho\theta, \rho^3\theta, \rho^5\theta, \rho^7\theta, \varepsilon\}, \{\rho^4, \theta, \rho^4\theta, \varepsilon\}, \{\rho^4, \rho\theta, \rho^5\theta, \varepsilon\}$, $\{\rho^4, \rho^2\theta, \rho^6\theta, \varepsilon\}, \{\rho^4, \rho^3\theta, \rho^7\theta, \varepsilon\}, \{\theta, \varepsilon\}, \{\rho\theta, \varepsilon\}, \{\rho^2\theta, \varepsilon\}, \{\rho^3\theta, \varepsilon\}, \{\rho^4\theta, \varepsilon\}, \{\rho^5\theta, \varepsilon\}, \{\rho^6\theta, \varepsilon\}, and\ \{\rho^7\theta, \varepsilon\}$. Similar to the rotations, I will focus on the subgroups along with their reflections that are generated by $\rho, \rho^2, \rho^4, and\ \rho^8$. The subgroup $\{\rho^2, \rho^4, \rho^6, \theta, \rho^2\theta, \rho^4\theta, \rho^6\theta, \varepsilon\}$ is generated by $\rho^2\ and\ \theta$ and $\{\rho^2, \rho^4, \rho^6, \rho\theta, \rho^3\theta, \rho^5\theta, \rho^7\theta, \varepsilon\}$ is the subgroup that is generated by ρ^2 and $\rho\theta$. Notice that ρ^2 generates two subgroups that contain reflections. The subgroup $\{\rho^4, \theta, \rho^4\theta, \varepsilon\}$ is generated by $\rho^4\ and\ \theta$, $\{\rho^4, \rho\theta, \rho^5\theta, \varepsilon\}$ is the subgroup that is generated by $\rho^4\ and\ \rho\theta$, $\{\rho^4, \rho^2\theta, \rho^6\theta, \varepsilon\}$ is the subgroup that is generated by $\rho^4\ and\ \rho^2\theta$, and $\{\rho^4, \rho^3\theta, \rho^7\theta, \varepsilon\}$ is the subgroup that is generated by $\rho^4\ and\ \rho^3\theta$. Thus, ρ^4 will generate four subgroups that contain reflections. The remaining subgroups that contain reflections and the identity are $\{\theta, \varepsilon\}, \{\rho\theta, \varepsilon\}, \{\rho^2\theta, \varepsilon\}, \{\rho^3\theta, \varepsilon\}, \{\rho^4\theta, \varepsilon\}, \{\rho^5\theta, \varepsilon\}, \{\rho^6\theta, \varepsilon\}, and\ \{\rho^7\theta, \varepsilon\}$. Notice that ρ^8 will generate eight subgroups that contain reflections. This section demonstrates that D(8) is equivalent to 4+1+2+4+8 which results in 19 total subgroups. Therefore, the number of subgroups of D(8) is equal to the number of factors of eight plus each factor of eight.

Throughout this section I will refer to each lattice of D(n)'s subgroups to validate my conjecture S_n will represent the number of subgroups of D(n). The number of subroups for D(3) is represented as S_3. The collection of subgroups of D(n) demonstrates that S_3 is 6 and the factors of 3 are 1 and 3, then S_3 is 2+3+1, or 6 total subgroups. Similarly, D(5) has 8 subgroups, and my conjecture states that $S_5 =$ 2+1+5, or 8 total subgroups. The formula that determines the number of subgroups of D(n) is gleaned from the lattice of subgroups, as is reflected in the table below.

n	**Number of Subgroups of D(n)**	**S_n**
3	6	2+3+1
4	10	3+1+2+4
5	8	2+1+5
6	16	4+1+2+3+6
7	10	2+1+7
8	19	4+1+2+4+8

As stated earlier, the symmetries of any regular n-sided polygons are the elements of D(n), and the subgroups of D(n). The function which determines the number of subgroups of D(n) will utilize τ and σ. By definition, "Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n, and $\sigma(n)$ denote the sum of these divisors" (Burton, 1976). For example, the number 14 has the positive divisors 1, 2, 7, and 14 which means $\tau(14)=4$ and $\sigma(14) = 1 + 2 + 7 + 14 = 24$; consequently, S_{14} is equal to four plus twenty four.

The reader's reflection should be, "Does this formula work for every dihedral group D(n)?"

The Fundamental Theorem of Arithmetic states, If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ is the prime factorization of n > 1, then the positive divisors of n are precisely those integers d of the form $d = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, where $0 \le a_i \le k_i$ (i=1,2,…, r) stands. This implies that if d is a divisor of n, then d will generate the subgroup of rotations ρ^d, ρ^{2d}, …, ε, a subgroup of D(n). From this we can determine the number of subgroups of D(n). Let's begin by determining the value for S_{24}. The factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24. The prime factorization of 24 is $2^3 3^1$; thus, 24 has (3+1)(1+1) = 8 positive factors, implying that D(24) has eight subgroups that contain only rotations. The sum of the factors is $1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 56 = \sigma(24)$, implying that D(24) has 56 subgroups that contain reflections; therefore, $S_{24} = \tau(24) + \sigma(24) = 8 + 56 = 64$. D(24) has 64 subgroups.

Proof

Now that we have an understanding of how each subgroup is generated, and know that the formula works for S_3 through S_8, I will prove that $S_n = \tau(n) + \sigma(n)$ for any given n. The proof for this consists of two parts: a) prove that $\tau(n)$ represents the number of subgroups that only contain rotations and b) prove that $\sigma(n)$ represents the number of subgroups that contain reflections.

a. By definition, $\tau(n)$ denotes the number of positive divisors of a positive integer n. Let d and n represent positive integers such that d is a divisor of n. Since d is a divisor of n then there exists an m such that m=n/d. ρ^d will generate the closed set $\{\rho^d, \rho^{2d}, \rho^{3d}, \dots, \rho^{n-d}, \rho^n\}$ of rotations. In order for this closed set generated by ρ^d to be a subgroup, it must contain the inverse for every element in the set using properties of exponents, $\rho^d \cdot \rho^{n-d} = \rho^n = \varepsilon, \rho^{2d} \cdot \rho^{n-2d} = \varepsilon, \rho^{3d} \cdot \rho^{n-3d} = \varepsilon, \cdots$. Because the set generated by ρ^d is closed and contains the inverse of each element, then the set generated by ρ^d is a subgroup. This demonstrates that every power of ρ which is a divisor of n will generate a subgroup of rotations. Also, any multiple of d that is not also a divisor of n will generate the same subgroup as ρ^d, and any power of ρ that is relatively prime to n will generate the same subgroup as ρ. In conclusion, the number of subgroups of D(n) that only contain rotations is equal to the number of divisors of n which can be symbolized by $\tau(n)$.

b. By definition, $\sigma(n)$ is the sum of the positive divisors of n. I want to prove that $\sigma(n)$ represents the number of subgroups that contain reflections. Let the variables z, n and d represent positive integers such that d is a divisor of n, $\sigma(n)=z+d$, and $0 \le a_i \le d$ (a_i=1,2,…, d). The subgroups generated by ρ^d and $\rho^{a_i}\theta$ can be listed as; $\{\rho^d, \rho^{2d}, \dots, \varepsilon, \theta, \rho^d\theta, \rho^{2d}\theta, \dots\}$, $\{\rho^d, \rho^{2d}, \dots, \varepsilon, \rho\theta, \rho^{d+1}\theta, \dots\}$, $\{\rho^d, \rho^{2d}, \dots, \varepsilon, \rho^2\theta, \rho^{d+2}\theta, \dots\}$, … , $\{\rho^d, \rho^{2d}, \dots, \varepsilon, \rho^{d-1}\theta, \rho^{2d-1}\theta, \dots\}$. Each subgroup generated by ρ^d and $\rho^{a_i}\theta$ will contain a

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specific element from the set $\{\theta, \rho\theta, \rho^2\theta, \dots, \rho^{d-1}\theta\}$. This set has a total of d elements which means that each ρ^d and $\rho^{a_i}\theta$ will generate d subgroups that contain reflections; therefore, the number of subgroups of D(n) that contain reflections is equal to the sum of the divisors of n. Since $\tau(n)$ represents the subgroups only containing rotations and $\sigma(n)$ represents the subgroups containing reflections, then $S_n = \tau(n) + \sigma(n)$ for any given n.

Conjugacy Classes in Sn

Let G be any group. Two elements α and σ of G are said to be conjugate if α = γσγ-1 for some γ∈G (Samaila, 2010). In other words, if σ, γ ∈G, we define the conjugate of σ by γ or σ by γ -1 to be the element γσγ-1 or γ -1σγ respectively. Proposition 1: Let G be a group, and define the relation ∼ on G by α∼σ if α and σ are conjugate in G. Then ∼ is an equivalence relation (Bianchi, 2001). Proof: All we need to do is to show that ∼ satisfies the three defining properties of an equivalence relation. 1. For α∈G, α∼α since eαe -1 = α, which shows that ∼ is reflexive. 2. Suppose α∼σ, then for some γ∈G, α=γσγ-1. Now, γ -1αγ = γ -1(γσγ-1)γ = (γ -1γ)σ(γ -1γ) = eσe = σ i.e. if we conjugate α by γ -1, then we have σ∼α. Thus, ∼ is symmetric. 3. Let α, β, σ∈G such that α∼β and β∼σ. Then α = γβγ-1 and β = μσμ-1 for some γ, μ∈G. Now α = γ(μσμ-1)γ -1 = (γμ)σ(μ -1γ -1) = (γμ)σ(γμ) -1. Hence, conjugating σ by (γμ) to get α means that α∼σ. i.e. ∼ is transitive. Hence, ∼ is an equivalence relation.∗ Since the relation ∼ is an equivalence relation on G, its equivalence classes partition G. The equivalence classes under this relation are called the conjugacy classes of G. Hence the conjugacy class of α∈G is given by [α] = {γαγ-1|γ∈G}.

Representation of the Conjugacy Classes in Dn

Considering the definition of Conjugacy class explained above, if we represent the elements of Dn as {I, α, α 2 , …, α n-1 ,α i β; 0 ≤ i ≤ n-1} where each element is to represent a conjugacy class, then we shall have the size of the conjugacy classes. Again, there are (n-1)/2 pairs of conjugate rotations when n is odd (exclude the identity) and (n-2)/2 pairs of conjugate rotations for even n (exclude the identity and α n/2). In both cases, whether n is even or odd, the sum of the sizes of the conjugacy classes in Dn equals 2n. Reprt. I α α 2 … α (n-1)/2 β Size 1 2 2 … 2 n Table 1: Conjugacy class representation in Dn for n odd Reprt. I α α 2 … α (n-2)/2 α n/2 β αβ Size 1 2 2 … 2 1 n/2 n/2 Table 2: Conjugacy class representation in Dn for n even Obviously, in table 1 and 2 of the representations of the conjugacy classes of Dn above, the sum of the sizes of the conjugacy classes amounted to 2n.

Center of Dn

Recall that the centralizer of the subgroup H in a group Dn is the set of elements of Dn which commute with every elements of H, namely C(H)Dn = {g∈Dn | αg = gα for all α ∈ H}. Hence, the centralizer of the subgroup H of the group Dn is the subgroup H itself if H represent the set of all rotations (including identity) in Dn. Again the center of the group Dn is the subgroup of Dn defined by Z(Dn) = {g∈Dn : gh = hg ∀

h∈Dn}. Thus, Z(Dn) = {I}, the trivial subgroup if n is odd and Z(Dn) = {I, α n/2 } if n is even. The center of any group G is a normal subgroup of e proposition are satisfied.

The Class Equation of a Finite Group

Given any finite group G, let Z(G) be the center of G. Then G∩{Z(G)}c is a disjoint union of conjugacy classes. Let m be the number of conjugacy classes contained in G∩{Z(G)}c , and let i1, i2, …, im be the number of elements in these conjugacy classes. Then ij > 1 for all j, since the centre Z(G) of G is the subgroup of G consisting of those elements of G whose conjugacy class contains just one element, see tables 1 and 2 above. Now the group G is the disjoint union of its Conjugacy classes, and therefore, |G| = |Z(G)| + i1 + i2 + … + im

Summary

The group structures of Sn and Dn were examined and their Conjugacy classes. It was found that all conjugate elements in Sn have the same cycle type, i.e. if α,σ∈Sn such that α is conjugate to σ, then l(α) = l(σ). While in Dn, its elements are partitioned in to two disjoint sets (one consists of rotations and the other for the reflections) of the same order (Samaila, 2010). Each rotation is conjugate to its inverse, noting that for the identity element I and the rotation α n/2 (for n-even), each is conjugate to itself. For n-odd, the reflection β is conjugate to every other reflections while for n- even, β is conjugate to half of the reflections while the reflection αβ is conjugate to the remaining half of the reflections. We have also seen that the relation “Conjugacy” is an equivalence relation. The center of Dn is found to be the trivial subgroup {I} when n is odd and {I, α n/2 } when n is even. And finally, two class equations for Dn were derived.

REFERENCES

[1] A. Beltran and M.J. Felipe, Some class size conditions implying solvability of finite groups, J. Group Theory, 9(6) (2006), 787-797

[2] A. Beltran and M.J. Felipe, Structure of finite groups under certain arithmetical conditions on class sizes, J. Algebra, 319(3) (2008), 897-910

[3] D. Samaila, I.B. Mshelia and M.S. Adamu, The symphonic character of dicyclic group Q12 of order 24 using group action and topology, JOLORN, 11(2) (2010), 39-44.

[4] D. Samaila, I.B. Mshelia and M.S. Adamu, A derived model for the construction of double dihedral groups Q2N and their properties, JOLORN, 11(2) (2010), 115-123.

[5] S. Lang, Undergraduate Algebra (3rd ed.), Springer-Verlag, Berlin, New York, 2005.

[6] Z. Arad, M. Muzychuk and A. Oliver, On groups with conjugacy classes of distinct sizes, J. Algebra, 280(2) (2004), 537-576.