Inequalities for Polynomials having t-fold zeros at origin.

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Inequality (1) is a well-known result of S.Bernstein (for reference see [5] or [14]), whereas inequality (2) is a simple deduction from maximum modulus principle (see [17]). In both (1) and (2) equality holds only when \( P(z) \) is a constant multiple of \( z^n \).

2010 Mathematics Subject Classification. 30C10, 30C15.
Keywords and Phrases. Inequalities, polynomials, derivative.

Abstract

In this paper we consider the class of polynomials

\[ P(z) = z^t \left( a_t + \sum_{j=\mu}^{n} a_j z^{j-t} \right), \]

\( t + 1 \leq \mu \leq n \) not vanishing in the disk \( |z| < k \), \( k \geq 1 \) except for \( t \)-fold zeros at origin. For \( k \geq 1 \), we investigate the dependence of \( \text{Max}_{|z|=1} |P(Rz) - R^k P(z)| \)
and \( \text{Max}_{|z|=1} |P(Rz) - P(z)| \) on \( \text{Max}_{|z|=1} |P(z)| \), we also estimate \( \text{Max}_{|z|=R} |P'(z)| \)
in terms of \( \text{Max}_{|z|=r} |P'(z)| \), \( 0 \leq r \leq R \leq k \). Our results not only generalize some polynomial inequalities but also a variety of results can be deduced from it by a fairly uniform procedure.

Introduction and Statement of Results.

Let \( P(z) \) be a polynomial of degree \( n \) and \( P'(z) \) its derivative, then

\[ \text{Max}_{|z|=1} |P'(z)| \leq n \text{Max}_{|z|=1} |P(z)| \quad (1) \]

\[ \text{Max}_{|z|=R>1} |P(z)| \leq R^n \text{Max}_{|z|=1} |P(z)| \quad (2) \]
If we restrict ourselves to a class of polynomials of degree $n$ having no zeros in $|z| < 1$, then

$$\text{Max}_{|z|=1}|P'(z)| \leq \frac{n}{2} \text{Max}_{|z|=1}|P(z)|$$  \hspace{1cm} (3)

$$\text{Max}_{|z|=R>1}|P(z)| \leq \frac{R^n + 1}{2} \text{Max}_{|z|=1}|P(z)|$$  \hspace{1cm} (4)

Inequality (3) was conjectured by Erdős and later verified by Lax [12], where as Ankeny and Rivlin [1] used (3) to prove (4).

As an extension of (3) Malik [13] verified that if $P(z)$ does not vanish in $|z| < k$, $k \geq 1$, then

$$\text{Max}_{|z|=1}|P'(z)| \leq \frac{n}{1 + k} \text{Max}_{|z|=1}|P(z)|$$  \hspace{1cm} (5)

Chan and Malik [6] generalized (5) in a different direction and proved that if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$ is a polynomial of degree $n$ which does not vanish in $|z| < k$, $k \geq 1$, then

$$\text{Max}_{|z|=1}|P'(z)| \leq \frac{n}{1 + k^\mu} \text{Max}_{|z|=1}|P(z)|$$  \hspace{1cm} (6)

Inequality (6) was independently proved by Qazi [16, Lemma 1], who also under the same hypothesis proved that

$$\text{Max}_{|z|=1}|P'(z)| \leq n \left( 1 + \frac{\mu}{n} \frac{|a_n|}{|a_0|} \frac{k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n} \frac{|a_n|}{|a_0|} (k^{\mu+1} + k^{2\mu})} \text{Max}_{|z|=1}|P(z)| \right)$$  \hspace{1cm} (7)

Recently, Aziz and Shah [4] investigated the dependence of $\text{Max}_{|z|=1}|P(Rz) - P(z)|$ on $\text{Max}_{|z|=1}|P(z)|$, where $R > 1$ and proved the following results.

**Theorem A.** Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ be a polynomial of degree $n$ which does not vanish in $|z| < k$ where $k \geq 1$ the for every $R \geq 1$ and $|z| = 1$

$$|P(Rz) - P(z)| \leq (R^n - 1) \frac{1 + \left( \frac{R^n - 1}{R^n - 1} \right) \frac{|a_n|}{|a_0|} k^{\mu+1}}{1 + k^{\mu+1} + \left( \frac{R^n - 1}{R^n - 1} \right) \frac{|a_n|}{|a_0|} (k^{\mu+1} + k^{2\mu})} \text{Max}_{|z|=1}|P(z)|$$  \hspace{1cm} (8)

they also proved the following improvement and generalization of a result due to Bidkham and Dewan [9].
Theorem B. If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ is a polynomial of degree $n$ having no zeros in the disk $|z| \leq k$, $k \geq 0$, then for $0 \leq r \leq R \leq k$

$$Max_{|z|=r}|P'(z)| \leq \frac{nR^{n-1}(R^\mu + k^\mu)^{n-1}}{(r^\mu + k^\mu)^{n-1}} \left( Max_{|z|=r}|P(z)| - Min_{|z|=r}|P(z)| \right)$$

(9)

The result is best possible and equality holds for the polynomial $P(z) = (z^\mu + k^\mu)^{n-1}$ where $n$ is a multiple of $\mu$.

In this paper we consider the class of polynomials $P(z) = z^t(a_1 + \sum_{j=\mu}^{n} a_j z^{j-t})$, $t+1 \leq \mu \leq n$, not vanishing in the disk $|z| < k$, $k \geq 1$, except at $t$-fold zeros at origin and investigate the dependence of $Max_{|z|=1}|P(Rz) - R^t P(z)|$ and $Max_{|z|=1}|P(Rz) - P(z)|$ on $Max_{|z|=1}|P(z)|$ and $Max_{|z|=r}|P'(z)|$, $0 \leq r \leq R \leq k$.

We shall first present the following generalization of Theorem A as a special case which also provides a variety of results.

Theorem 1.1. Let $P(z) = z^t(a_1 + \sum_{j=\mu}^{n} a_j z^{j-t})$, $t+1 \leq \mu \leq n$, be a polynomial of degree $n$, which does not vanish in $|z| \leq k$, $k \geq 1$ except for $t$-fold zeros at origin then for every $R \geq 1$, $|z| = 1$

$$|P(Rz) - R^t P(z)| \leq (R^n - R^t)^{n-1} \left( \frac{1 + (R^{n-1} - 1)^{n-1}}{1 + k^{\mu+1} + \left( \frac{R^{n-1} - 1}{R^{n-1} - 1} \right)^\mu \left( k^{\mu+1} + k^{2\mu} \right)} \right) Max_{|z|=1}|P(z)|$$

(10)

Remark 1. If we take $t = 0$ in inequality (10), we get Theorem A.

The following result which also provides an interesting generalization of Theorem A can be easily deduced from Theorem 1.1.

Theorem 1.2. Let $P(z) = z^t(a_1 + \sum_{j=\mu}^{n} a_j z^{j-t})$, $t+1 \leq \mu \leq n$, be a polynomial of degree $n$, which does not vanish in $|z| \leq k$, $k \geq 1$ except for $t$-fold zeros at origin then for every $R \geq 1$, $|z| = 1$

$$|P(Rz) - P(z)| \leq$$

$$\left( (R^n - 1) + (R^n - R^t) \left( \frac{1 + (R^{n-1} - 1)^{n-1}}{1 + k^{\mu+1} + \left( \frac{R^{n-1} - 1}{R^{n-1} - 1} \right)^\mu \left( k^{\mu+1} + k^{2\mu} \right)} \right) \right) Max_{|z|=1}|P(z)|$$

(11)

Remark 2. If we take $t = 0$ in inequality (11), we get inequality (8)

If we use the fact that $|P(Rz)| \leq |P(Rz) - P(z)| + |P(z)|$, then the following corollary is an immediate consequence of Theorem 1.2.
Corollary 1.1. Let \( P(z) = z^t(a_t + \sum_{j=\mu}^{n} a_j z^{j-t}) \), \( t + 1 \leq \mu \leq n \), be a polynomial of degree \( n \), which does not vanish in \( |z| \leq k \), \( k \geq 1 \) except for \( t \)-fold zeros at origin then for every \( R \geq 1 \),
\[
\frac{R^n}{1 + k^{\mu+1}} \leq \frac{\text{Max}_{|z|=R} |P(z)|}{\text{Max}_{|z|=1} |P(z)|}.
\]
(12)

The inequality
\[
\frac{R^\mu - 1}{R^{n-t} - 1} \leq \frac{\mu}{n-t}
\]
holds for all \( R \geq 1 \) and \( t + 1 \leq \mu \leq n \). To prove this inequality we observe that it is trivial for \( R = 1 \) and for \( R \geq 1 \) it easily follows when \( \mu = n - t \). Hence to establish \( (13) \), it suffices to consider the case \( t + 1 \leq \mu \leq n - 1 \) and \( R > 1 \). Now if \( R > 1 \) and \( t + 1 \leq \mu \leq n - 1 \), then we have
\[
\mu R^{n-t} - (n-t) R^n + (n - \mu - t) = \mu R^{n-t} \left( R^n R^{\mu - 1} - (n - \mu - t) \right) = (R-1) \left\{ \mu R^{n-t-\mu-1} + R^{n-t-\mu-2} + \ldots + 1 \right\}
\]
\[
\geq (R-1) \left\{ \frac{\mu(n-t)}{(R-1)^2} \right\}
\]
This implies \( \mu(R^{n-t}-1) \geq (n-t)(R^{\mu}-1) \) for all values of \( R > 1 \) and \( 1+t \leq \mu \leq n-1 \), which is equivalent to \( (13) \) with the help of inequality \( (13) \), a simple consequence yields.
\[
R^n \left( 1 + \left( \frac{R^n-1}{R^n-1} \right)^{|\frac{a_\mu}{a_t}|} |k^{\mu+1}| + k^{\mu+1} + \left( \frac{R^n-1}{R^n-1} \right)^{|\frac{a_\mu}{a_t}|} |k^{2\mu}| \right)
\]
\[
1 + k^{\mu+1} + \left( \frac{R^n-1}{R^n-1} \right)^{|\frac{a_\mu}{a_t}|} (k^{2\mu} + k^{\mu+1})
\]

\[
\leq \frac{R^n \left( 1 + \frac{\mu}{n-\mu} \left| a_\mu \right| k^{\mu+1} \right) + k^{\mu+1} + \frac{\mu}{n-\mu} \left| a_\mu \right| k^{2\mu}}{1 + k^{\mu+1} + \frac{\mu}{n-\mu} \left| a_\mu \right| (k^{2\mu} + k^{\mu+1})}.
\]  

(14)

Hence from Theorem 1.2, we easily deduce the following.

**Corollary 1.2.** If \( P(z) = z^{t}(a_t - \sum_{j=\mu}^{n} a_j z^{j-t}) \), \( t + 1 \leq \mu \leq n \), is a polynomial of degree \( n \), which does not vanish in \( |z| \leq k \), \( k \geq 1 \) except for \( t \)-fold zeros at origin then for every \( R > 1 \),

\[
\max_{|z|=R>1} |P(z)| \leq \frac{R^n \left( 1 + \frac{\mu}{n-\mu} \left| a_\mu \right| k^{\mu+1} \right) + k^{\mu+1} + \frac{\mu}{n-\mu} \left| a_\mu \right| k^{2\mu}}{1 + k^{\mu+1} + \frac{\mu}{n-\mu} \left| a_\mu \right| (k^{2\mu} + k^{\mu+1})} \max_{|z|=1} |P(z)|.
\]  

(15)

Inequality (12) provides a refinement of a result due to Govil and Dewan ([8], Theorem 1.9) which is also a special case of inequality (15) when \( \mu = t + 1 \). Next if we take \( \mu = t + 1 \) in Theorem 1.2, we get

**Corollary 1.3.** Let \( P(z) = z^{t}(a_t + \sum_{j=\mu}^{n} a_j z^{j-t}) \), \( t + 1 \leq \mu \leq n \), be a polynomial of degree \( n \), which does not vanish in \( |z| \leq k \), \( k \geq 1 \) except for \( t \)-fold zeros at origin then for every \( R > 1 \),

\[
|P(Rz) - P(z)| \leq \]
\[ \left( \frac{1}{(R^t-1)}+\frac{1}{(R^n-R^t)} \right) \left( \frac{1 + \left( \frac{R^t-1}{R^n-1} \right)}{1 + \left( \frac{R^t+1}{R^n-1} \right)} \right) \text{Max}_{|z|=1} |P(z)|. \] (16)

Taking \( t = 1 \), in (16), we get inequality (14) of Aziz and Shah [4].

**Remark 3.** Dividing the two sides of inequality (16) by \( R-1 \) and making \( R \to 1 \), it follows that, if \( P(z) = z^t(a_t + \sum_{j=t}^{n} a_j z^{j-t}) \), \( t+1 \leq \mu \leq n \), be a polynomial of degree \( n \), with \( t \)-fold zeros at origin and \( P(z) \neq 0 \) in \( |z| < k, k \geq 1 \), then

\[ |P'(z)| \leq \frac{t+(n-t)}{1+k^{t+2}+(\frac{t+1}{n-t})^2 a_{t+1} \left( k^{t+2} + k^{\mu+1} \right)} \text{Max}_{|z|=1} |P(z)|. \] (17)

Inequality (17) is a refinement of inequality (5) and for \( t = 0 \) it reduces to inequality (15) of Aziz and Shah [4] which is a refinement of inequality (5) and was independently proved by Govil, Rahman and Schmeisser [11]. Using (13) and the fact \( \frac{\mu}{n-t} a_{\mu} a_t \leq 1 \) it can be easily verified that

\[ \frac{1 + \left( \frac{R^{t-1}}{R^n-1} \right)}{1 + k^{\mu+1} + \left( \frac{R^{t-1}}{R^n-1} \right) a_{t+1} \left( k^{2\mu} + k^{\mu+1} \right)} \leq \frac{1}{1 + k^{\mu}}. \] (18)

By using these observations, the following result is an immediate consequence of Theorem 1.1.

**Corollary 1.4.** If \( P(z) = z^t(a_t + \sum_{j=t}^{n} a_j z^{j-t}) \), \( t+1 \leq \mu \leq n \), is a polynomial of degree \( n \), which does not vanish in \( |z| < k, k \geq 1 \) except for \( t \)-fold zeros at origin then for every \( R \geq 1 \)

\[ |P(Rz) - R^t P(z)| \leq \frac{R^n - R^t}{1 + k^{\mu}} \text{Max}_{|z|=1} |P(z)| \] (19)

and it follows that

\[ \text{Max}_{|z|=R} |P(z)| \leq \frac{R^n + k^{\mu}}{1 + k^{\mu}} \text{Max}_{|z|=1} |P(z)| \] (20)

Inequality (20) is a generalization of a result due to Govil and Datt [7, Theorem 1.6]. Also for \( t = 0, k = \mu = 1 \) inequality (20) reduces to inequality (4) due to Ankeny and Rivlin.

Next we shall present the following generalization of Theorem B.

**Theorem 1.3.** If \( P(z) = z^t(a_t + \sum_{j=t}^{n} a_j z^{j-t}) \), \( t+1 \leq \mu \leq n \), is a polynomial of degree \( n \), having no zero in the disk \( |z| < k, k \geq 0 \) except for \( t \)-fold zeros at origin then for \( 0 \leq r \leq R \geq k \),
\[ \text{Max}_{|z|=R} |P'(z)| \leq \frac{(n-t)R^{n-1}}{R^n+k^\mu} \cdot \left( \frac{R^{n-t}+k^\mu}{(r^n+k^\mu)} \right)^{\frac{n-t}{n}} \left\{ \frac{R^t}{t} \text{Max}_{|z|=r} |P(z)| \right\} + \frac{t}{R} \text{Max}_{|z|=R} |P(z)|. \] 

The result is best possible and equality holds for the polynomial \( P(z) = (z^n+k^\mu)^{\frac{n}{n-t}} \) where \( n \) is a multiple of \( \mu \).

**Remark 4.** Taking \( t=0 \) in Theorem 1.3 we get Theorem B.

If we take \( \mu = t+1, r=1 \) in Theorem 1.3, we get the following result.

**Corollary 1.5.** If \( P(z) = z^t \sum_{j=t+1}^n a_jz^j \) is a polynomial of degree \( n \), having no zero in the disk \( |z| < k, k \leq 1 \) except for \( t \)-fold zeros at origin then for \( 1 \leq R \leq k \),

\[ \text{Max}_{|z|=R} |P'(z)| \leq \frac{(n-t)R^{t+1}}{R^{t+1}+k^{t+1}} \cdot \left( \frac{R^{t+1}+k^{t+1}}{1+k^{t+1}} \right)^{\frac{n-t}{n}} \left\{ \frac{R^t}{t} \text{Max}_{|z|=1} |P(z)| \right\} + \frac{t}{R} \text{Max}_{|z|=R} |P(z)|. \] 

The result is sharp and equality holds for \( P(z) = (z^n+k^{t+1})^{\frac{n}{n-t}}, \mu = t+1 \).

If we take \( R = k = 1 \) in corollary 1.5 we get the following generalization of a result due to Aziz and Dawood [3]

\[ \text{Max}_{|z|=1} |P'(z)| \leq \frac{n+t}{2} \text{Max}_{|z|=1} |P(z)| - \frac{n-t}{2} \text{Min}_{|z|=1} |P(z)|. \]

The result is best possible for \( P(z) = (z+k)^n \).

If we take \( R = k = 1 \) in Theorem 1.3, we get the following generalization of a result due to Aziz and Shah [4, Cor. 6].

**Corollary 1.6.** If \( P(z) = z^t(a_t + \sum_{j=t}^n a_jz^{j-t}) \), \( t+1 \leq \mu \leq n \), is a polynomial of degree \( n \), having no zero in the disk \( |z| < 1 \), except for \( t \)-fold zeros at origin then for \( 0 < r \leq 1 \),

\[ \text{Max}_{|z|=1} |P'(z)| \leq \frac{n-t}{2} \left( \frac{1}{1+r^\mu} \right)^{\frac{n-t}{n}} \left\{ \frac{1}{r} \text{Max}_{|z|=r} |P(z)| \right\} + \text{Min}_{|z|=1} |P(z)|. \] 

The result is best possible and equality holds for the polynomial \( P(z) = (z^n+1)^{\frac{n}{n-t}} \) where \( n \) is a multiple of \( \mu \).
Lemmas

For the proofs of these Theorems we need the following Lemmas. The following Lemma is due to Aziz and Shah [4].

Lemma 1. If \( P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \) is a polynomial of degree \( n \) having no zeros in the disk \( |z| \leq k, k \geq 1 \) then for every \( R > 1 \) and \( |z| = 1 \)

\[
|P(Rz) - P(z)| \leq \frac{1}{k^{\mu+1}} \left\{ 1 + \frac{R^{\mu}}{R^{\mu+1}} \left| \frac{a_0}{a_0} \right| |k^{\mu+1} + 1 \right\} |Q(Rz) - Q(z)|
\]  (23)

Lemma 2. If \( P(z) \) is a polynomial of degree \( n \) Then for every \( R > 1 \)

\[
|P(Rz) - P(z)| + |Q(Rz) - Q(z)| \leq (R^\mu - 1) Max_{|z|=1}|P(z)|
\]  (24)

The above Lemma due to Aziz, [2] (see also [10]).

Lemma 3. If \( P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \) is a polynomial of degree \( n \) having no zeros in the disk \( |z| < k, k \geq 1 \) then

\[
Max_{|z|=1}|P'(z)| \leq \frac{1}{1 + k^{\mu}} \left\{ Max_{|z|=1}|P(z)| - Min_{|z|=k}|P(z)| \right\}
\]  (25)

The Lemma was proved by Dewan and Pukhta, [15].

Lemma 4. Let \( P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \) be a polynomial of degree \( n \) such that

\[
M(P, r) = Max_{|z|=r}|P(z)| \quad \text{and} \quad m(P, r) = Min_{|z|=r}|P(z)|.
\]

If \( P(z) \) has no zeros in \( |z| < k, k > 0 \) then for \( 0 \leq r \leq R \leq k \)

\[
M(P, r) \geq \left( \frac{r^\mu + k^\mu}{R^\mu + k^\mu} \right)^{\frac{n}{\mu}} M(P, R) + \left\{ 1 - \left( \frac{r^\mu + k^\mu}{R^\mu + k^\mu} \right)^{\frac{n}{\mu}} \right\} m(P, k).
\]  (26)

The result is sharp and equality holds for the polynomial \( P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}} \) where \( n \) is multiple of \( \mu \).

Lemma 4 is due to Aziz and Shah [4].
Proofs of Theorems

Proof of Theorem 1.1. Since \( P(z) = z^t(a_t + \sum_{j=\mu}^{n} a_j z^{j-t}) = z^t H(z) \), \( t + 1 \leq \mu \leq n \) does not vanish in \( |z| < k \), \( k \geq 1 \) except for \( t \)-fold zeros at origin. Applying Lemma 1 to the polynomial \( H(z) \) of degree \( n - t \), we get,

\[
\frac{k^{\mu+1} \left( \frac{R^\mu - 1}{R^{n-t-1} - 1} \right) \frac{a_\mu}{a_t} k^{\mu-1} + 1}{1 + \frac{R^\mu - 1}{R^{n-t-1} - 1} \frac{a_\mu}{a_t} k^{\mu+1}} |H(Rz) - H(z)| \leq |G(Rz) - G(z)|
\]  (27)

Where,

\[
G(z) = z^{n-t} H\left(\frac{1}{z}\right).
\]

Inequality (27) with the help of Lemma 2 implies that

\[
\left\{ 1 + \frac{k^{\mu+1} \left( \frac{R^\mu - 1}{R^{n-t-1} - 1} \frac{a_\mu}{a_t} k^{\mu-1} \right) + 1}{1 + \frac{R^\mu - 1}{R^{n-t-1} - 1} \frac{a_\mu}{a_t} k^{\mu+1}} \right\} |H(Rz) - H(z)|
\]

\[
\leq |H(Rz) - H(z)| + |G(Rz) - G(z)|
\]

\[
\leq (R^{n-t} - 1) \max_{|z|=1} |H(z)|.
\]
This gives

\[ |H(Rz) - H(z)| \leq (R^{n-t} - 1) \left( 1 + \frac{Re^{s-1}}{Re^{s-1} - 1} \frac{a_{\mu}}{a_t} |k^{\mu+1}| \right) \max_{|z|=1} |H(z)| \]

or

\[ |R^t z^t H(Rz) - R^t z^t H(z)| \leq (R^n - R^t) \left( 1 + \frac{Re^{s-1}}{Re^{s-1} - 1} \frac{a_{\mu}}{a_t} |k^{\mu+1}| \right) \max_{|z|=1} |H(z)|. \]

This gives,

\[ |P(Rz) - R^t P(z)| \leq (R^n - R^t) \left( 1 + \frac{Re^{s-1}}{Re^{s-1} - 1} \frac{a_{\mu}}{a_t} |k^{\mu+1}| \right) \max_{|z|=1} |P(z)| \]

which is inequality (10) and this proves Theorem 1.1 completely.

Proof of Theorem 1.2. From inequality (10) it follows that

\[ |P(Rz) - P(z) + P(z) - R^t P(z)| \leq (R^n - R^t) \left( 1 + \frac{Re^{s-1}}{Re^{s-1} - 1} \frac{a_{\mu}}{a_t} |k^{\mu+1}| \right) \max_{|z|=1} |P(z)| \]

\[ |P(Rz) - P(z)| \leq \left( R^t - 1 + (R^n - R^t) \left( 1 + \frac{Re^{s-1}}{Re^{s-1} - 1} \frac{a_{\mu}}{a_t} |k^{\mu+1}| \right) \right) \max_{|z|=1} |P(z)| \]

which is inequality (11) and hence Theorem 1.2 is proved.

Proof of Theorem 1.3. By hypothesis \( P(z) = z^t (a_t + \sum_{j=1}^{n} a_j z^{j-t}) = z^t H(z) \), \( t + 1 \leq \mu \leq n \) does not vanish in \( |z| < k, k \geq 1 \) except for t-fold zeros at origin, therefore the polynomial \( F(z) = H(Rz) \) has no zeros in \( |z| < \frac{k}{R}, \frac{k}{R} \geq 1 \) Applying Lemma 3 to the polynomial \( F(z) \) we get,

we get

\[ |F'(z)| \leq \frac{n-t}{1 + \frac{k^\mu}{R^t}} \left( \max_{|z|=1} |F(z)| - \min_{|z|=\frac{k}{R}} |F(z)| \right). \]

which gives,

\[ \max_{|z|=R} |H'(z)| \leq \frac{(n-t)R^{t-1}}{R^t + k^\mu} \left\{ \max_{|z|=R} |H(z)| - \min_{|z|=k} |H(z)| \right\} \]
Now if \(0 \leq r \leq R \leq k\), then by Lemma 4 we have,

\[
\max_{|z|=R}|H(z)| \leq \left(\frac{r^\mu + k^\mu}{R^\mu + k^\mu}\right)\frac{\frac{n-r}{\mu}}{\frac{n-k}{\mu}} \max_{|z|=r}|H(z)|
\]

\[
+ \left\{1 - \left(\frac{r^\mu + k^\mu}{R^\mu + k^\mu}\right)^{\frac{n-r}{\mu}}\right\}\min_{|z|=k}|H(z)|
\]

(29)

From (28) and (29) it follows that

\[
\max_{|z|=R}|H'(z)| \leq \frac{(n-t)R^{\mu-1}}{R^\mu + k^\mu} \left\{ \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n-r}{\mu}} \max_{|z|=r}|H(z)|
\]

\[
- \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n-r}{\mu}} \min_{|z|=k}|H(z)| \right\}
\]

\[
= \frac{(n-t)R^{\mu-1}}{R^\mu + k^\mu} \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n-r}{\mu}} \left\{ \max_{|z|=r}|H(z)| - \min_{|z|=k}|H(z)| \right\}
\]

since

\[
P'(z) = z^r H'(z) + tz^{r-1} H(z)
\]

\[
\max_{|z|=R}|P'(z)| = \max_{|z|=R}|z^r H'(z) + tz^{r-1} H(z)|
\]

\[
\leq R^r \max_{|z|=R}|H'(z)| + tR^{r-1} \max_{|z|=R}|H(z)|
\]

\[
= R^r \left(\frac{(n-t)R^{\mu-1}}{R^\mu + k^\mu} \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n-r}{\mu}} \left\{ \max_{|z|=r}|H(z)| - \min_{|z|=k}|H(z)| \right\}\right)
\]

\[
+ \frac{t}{R} R^r \max_{|z|=R}|H(z)|
\]
2693 | P a g e

which is equivalent to inequality (21) and this completes the proof of Theorem 1.3.

REFERENCES