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Inequalities for Polynomials having t-fold zeros at origin.

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Inequality (1) is a well-known result of S.Bernstein (for reference see [5] or [14]),where as inequality (2) is a simple deduction from maximum modulus principle (see [17]).In both (1) and (2) equality holds only when $P(z)$ is a constant multiple of z^n.

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Abstract

In this paper we consider the class of polynomials $P(z) = z^t(a_t + \sum_{j=\mu}^{n} a_j z^{j-t})$, $t+1 \leq \mu \leq n$ *not vanishing in the disk* $|z| < k, k \geq 1$ *except for t-fold zeros at origin.For* $k \geq 1$, *we investigate the dependence of* $Max_{|z|=1}|P(Rz) - R^t P(z)|$ *and* $Max_{|z|=1}|P(Rz) - P(z)|$ *on* $Max_{|z|=1}|P(z)|$, *we also estimate* $Max_{|z|=R}|P'(z)|$ *in terms of* $Max_{|z|=r}|P'(z)|, 0 \leq r \leq R \leq k$. *Our results not only generalize some polynomial inequalities but also a variety of results can be deduced from it by a fairly uniform procedure.*

Introduction and Statement of Results.

Let $P(z)$ be a polynomial of degree n and $P'(z)$ its derivative,then

$$Max_{|z|=1}|P'(z)| \leq nMax_{|z|=1}|P(z)| \qquad (1)$$

$$Max_{|z|=R>1}|P(z)| \leq R^n Max_{|z|=1}|P(z)| \qquad (2)$$

If we restrict ourselves to a class of polynomials of degree n having no zeros in $|z| < 1$, then

$$Max_{|z|=1}|P'(z)| \leq \frac{n}{2} Max_{|z|=1}|P(z)| \qquad (3)$$

$$Max_{|z|=R>1}|P(z)| \leq \frac{R^n + 1|}{2} Max_{|z|=1}|P(z)| \qquad (4)$$

Inequality (3) was conjectured by Erdös and later verified by Lax [12], where as Ankeny and Rivilin [1] used (3) to prove (4).

As an extension of (3) Malik [13] verified that if $P(z)$ does not vanish in $|z| < k, k \geq 1$, then

$$Max_{|z|=1}|P'(z)| \leq \frac{n}{1+k} Max_{|z|=1}|P(z)| \qquad (5)$$

Chan and Malik [6] generalized (5) in a different direction and proved that if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n$ is a polynomial of degree n which does not vanish in $|z| < k, k \geq 1$, then

$$Max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^\mu} Max_{|z|=1}|P(z)| \qquad (6)$$

Inequality (6) was independently proved by Qazi [16, Lemma 1],who also under the same hypothesis proved that

$$Max_{|z|=1}|P'(z)| \leq n \left(\frac{1 + \frac{\mu}{n}|\frac{a_n}{a_0}|k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n}|\frac{a_\mu}{a_0}|(k^{\mu+1} + k^{2\mu})} Max_{|z|=1}|P(z)| \right) \qquad (7)$$

Recently, Aziz and Shah [4] investigated the dependence of $Max_{|z|=1}|P(Rz) - P(z)|$ on $Max_{|z|=1}|P(z)|$, where $R > 1$ and proved the following results.

Theorem A. Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ be a polynomial of degree n which does not vanish in $|z| < k$ where $k \geq 1$ the for every $R \geq 1$ and $|z| = 1$

$$|P(Rz) - P(z)| \leq (R^n - 1) \frac{1 + (\frac{R^\mu - 1}{R^n - 1})|\frac{a_\mu}{a_0}|k^{\mu+1}}{1 + k^{\mu+1} + (\frac{R^\mu - 1}{R^n - 1})|\frac{a_\mu}{a_0}|(k^{\mu+1} + k^{2\mu})} Max_{|z|=1}|P(z)| \qquad (8)$$

they also proved the following improvement and generalization of a result due to Bidkham and Dewan [9].

Theorem B. If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ is a polynomial of degree n having no zeros in the disk $|z| \leq k, k \geq 0$, then for $0 \leq r \leq R \leq k$

$$Max_{|z|=R}|P'(z)| \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}\left\{Max_{|z|=r}|P(z)| - Min_{|z|=k}|P(z)|.\right\} \quad (9)$$

The result is best possible and equality holds for the polynomial $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ.

In this paper we consider the class of polynomials $P(z) = z^t(a_t + \sum_{j=\mu}^{n} a_j z^{j-t})$, $t+1 \leq \mu \leq n$, not vanishing in the disk $|z| < k, k \geq 1$, except at t-fold zeros at origin and investigate the dependence of $Max_{|z|=1}|P(Rz) - R^t P(z)|$ and $Max_{|z|=1}|P(Rz) - P(z)|$ on $Max_{|z|=1}|P(z)|$, we also estimate $Max_{|z|=R}|P'(z)|$ in terms of $Max_{|z|=r}|P'(z)|, 0 \leq r \leq R \leq k$.

We shall first present the following generalization of Theorem A as a special case which also provides a variety of results.

Theorem 1.1. Let $P(z) = z^t(a_t + \sum_{j=\mu}^{n} a_j z^{j-z}), t+1 \leq \mu \leq n$, be a polynomial of degree n, which does not vanish in $|z| \leq k, k \geq 1$ except for t-fold zeros at origin then for every $R \geq 1, |z| = 1$

$$|P(Rz) - R^t P(z)| \leq (R^n - R^t)\left\{\frac{1 + (\frac{R^\mu - 1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|k^{\mu+1}}{1 + k^{\mu+1} + (\frac{R^\mu - 1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|(k^{\mu+1} + k^{2\mu})}\right\}Max_{|z|=1}|P(z)| \quad (10)$$

Remark 1. If we take $t = 0$ in inequality(10), we get Theorem A.
The following result which also provides an interesting generalization of Theorem A can be easily deduced from Theorem 1.1.

Theorem 1.2. Let $P(z) = z^t(a_t + \sum_{j=\mu}^{n} a_j z^{j-z}), t+1 \leq \mu \leq n$, be a polynomial of degree n, which does not vanish in $|z| \leq k, k \geq 1$ except for t-fold zeros at origin then for every $R \geq 1, |z| = 1$

$$|P(Rz) - P(z)| \leq$$
$$\left\{(R^t - 1) + (R^n - R^t)\left(\frac{1 + (\frac{R^\mu - 1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|k^{\mu+1}}{1 + k^{\mu+1} + (\frac{R^\mu - 1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|(k^{\mu+1} + k^{2\mu})}\right)\right\}Max_{|z|=1}|P(z)| \quad (11)$$

Remark 2. If we take $t = 0$ in inequality (11), we get inequality (8)
If we use the fact that $|P(Rz)| \leq |P(Rz) - P(z)| + |P(z)|$, then the following corollary is an immediate consequence of Theorem 1.2.

Corollary 1.1. Let $P(z) = z^t(a_t + \sum_{j=\mu}^{n} a_j z^{j-t}), t+1 \leq \mu \leq n$, be a polynomial of degree n, which does not vanish in $|z| \leq k, k \geq 1$ except for t-fold zeros at origin then for every $R \geq 1$,

$$Max_{|z|=R>1}|P(z)| \leq \frac{R^n\left(1+(\frac{R^\mu-1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|k^{\mu+1}\right)+k^{\mu+1}+(\frac{R^\mu-1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|k^{2\mu}}{1+k^{\mu+1}+(\frac{R^\mu-1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})} Max_{|z|=1}|P(z)|. \quad (12)$$

The inequality

$$\frac{R^\mu - 1}{R^{n-t}-1} \leq \frac{\mu}{n-t} \quad (13)$$

holds for all $R \geq 1$ and $t+1 \leq \mu \leq n$.To prove this inequality we observe that it is trivial for $R = 1$ and for $R \geq 1$ it easily follows when $\mu = n-t$.Hence to establish (13),it suffices to consider the case $t+1 \leq \mu \leq n-1$ and $R > 1$.Now if $R > 1$ and $t+1 \leq \mu \leq n-1$, then we have

$$\begin{aligned}\mu R^{n-t} - (n-t)R^\mu + (n-\mu-t) &= \mu R^\mu(R^{n-t-\mu}-1) - (n-\mu-t)(R^\mu - 1)\\ &= (R-1)\Big\{\mu R^\mu(R^{n-t-\mu-1} + R^{n-t-\mu-2} + ... + 1)\\ &\quad - (n-t-\mu)(R^{\mu-1} + R^{\mu-2} + ... + 1)\Big\}\\ &\geq (R-1)\Big\{\mu(n-t-\mu)R^\mu - (n-t-\mu)\mu R^{\mu-1}\Big\}\\ &= \mu(n-t-\mu)(R-1)^2 > 0.\end{aligned}$$

This implies $\mu(R^{n-t}-1) \geq (n-t)(R^\mu - 1)$ for all values of $R > 1$ and $1+t \leq \mu \leq n-1$. which is equivalent to (13)
with the help of inequality (13),a simple consequence yields.

$$\frac{R^n\left(1+(\frac{R^\mu-1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|k^{\mu+1}\right)+k^{\mu+1}+(\frac{R^\mu-1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|k^{2\mu}}{1+k^{\mu+1}+(\frac{R^\mu-1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})}$$

$$\leq\frac{R^n\left(1+\frac{\mu}{n-t}|\frac{a_\mu}{a_t}|k^{\mu+1}\right)+k^{\mu+1}+\frac{\mu}{n-t}|\frac{a_\mu}{a_t}|k^{2\mu}}{1+k^{\mu+1}+\frac{\mu}{n-t}|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})}. \quad (14)$$

Hence from Theorem 1.2,we easily deduce the following.

Corollary 1.2. If $P(z)=z^t(a_j-\sum_{j=\mu}^{n}a_jz^{j-t}), t+1\leq\mu\leq n$, is a polynomial of degree n, which does not vanish in $|z|\leq k, k\geq 1$ except for t-fold zeros at origin

then for every $R>1$,

$$Max_{|z|=R>1}|P(z)|\leq\frac{R^n\left(1+\frac{\mu}{n-t}|\frac{a_\mu}{a_t}|k^{\mu+1}\right)+k^{\mu+1}+\frac{\mu}{n-t}|\frac{a_\mu}{a_t}|k^{2\mu}}{1+k^{\mu+1}+\frac{\mu}{n-t}|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})}Max_{|z|=1}|P(z)|. \quad (15)$$

Inequality (12) provides a refinement of a result deu to Govil and Dewan ([8],Theorem 1.9) which is also a special case of inequality (15) when $\mu=t+1$.Next if we take $\mu=t+1$ in Theorem 1.2,we get.

Corollary 1.3. Let $P(z)=z^t(a_t+\sum_{j=\mu}^{n}a_jz^{j-t}), t+1\leq\mu\leq n$, be a polynomial of degree n, which does not vanish in $|z|\leq k, k\geq 1$ except for t-fold zeros at origin then for every $R>1$,

$|P(Rz)-P(z)|\leq$

$$\left\{(R^t-1)+(R^n-R^t)\left(\frac{1+(\frac{R^{t+1}-1}{R^{n-t}-1})|\frac{a_{t+1}}{a_t}|k^{t+2}}{1+k^{t+2}+(\frac{R^{t+1}-1}{R^{n-t}-1})|\frac{a_{t+1}}{a_t}|(k^{2t+2}+k^{t+2})}\right)\right\}Max_{|z|=1}|P(z)|. \quad (16)$$

Taking $t=1$, in (16),we get inequality (14) of Aziz and Shah [4].

Remark 3. Dividing the two sides of inequality (16) by R-1 and making $R\to 1$, it follows that, if $P(z)=z^t(a_t+\sum_{j=\mu}^{n}a_jz^{j-t}), t+1\leq\mu\leq n$, be a polynomial of degree n, with t-fold zeros at origin and $P(z)\neq 0$ in $|z|<k, k\geq 1$, then

$$|P'(z)|\leq\frac{t+(n-t)\left(1+(\frac{t+1}{n-t})|\frac{a_{t+1}}{a_t}|k^{t+2}\right)}{1+k^{t+2}+(\frac{t+1}{n-t})|\frac{a_{t+1}}{a_t}|(k^{2t+2}+k^{t+2})}Max_{|z|=1}|P(z)|. \quad (17)$$

Inequality (17) is a refinement of inequality (5) and for $t=0$ it reduces to inequality (15) of Aziz and Shah [4] which is a refinement of inequality (5) and was independently proved by Govil,Rahman and Schmessier [11]. Using (13) and the fact $\frac{\mu}{n-t}|\frac{a_\mu}{a_t}|k^\mu\leq 1$ it can be easily verified that

$$\frac{1+(\frac{R^\mu-1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|k^{\mu+1}}{1+k^{\mu+1}+(\frac{R^\mu-1}{R^{n-t}-1})|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})}\leq\frac{1}{1+k^\mu}. \quad (18)$$

By using these observations, the following result is an immediate consequence of Theorem 1.1.

Corollary 1.4. If $P(z)=z^t(a_t+\sum_{j=\mu}^{n}a_jz^{j-t}), t+1\leq\mu\leq n$, is a polynomial of degree n, which does not vanish in $|z|<k, k\geq 1$ except for t-fold zeros at origin then for every $R\geq 1$

$$|P(Rz)-R^tP(z)|\leq\frac{R^n-R^t}{1+k^\mu}Max_{|z|=1}|P(z)| \quad (19)$$

and it follows that

$$Max_{|z|=R}|P(z)|\leq\frac{(R^n+k^\mu)+k^\mu(R^t-1)}{1+k^\mu}Max_{|z|=1}|P(z)| \quad (20)$$

Inequality (20) is a generalization of a result due to Govil and Datt [7, Theorem 1.6]. Also for $t=0, k=\mu=1$ inequality (20) reduces to inequality (4) due to Ankeny and Rivilin.

Next we shall present the following generalization of Theorem B.

Theorem 1.3. If $P(z)=z^t(a_t+\sum_{j=\mu}^{n}a_jz^{j-t}), t+1\leq\mu\leq n$, is a polynomial of degree n, having no zero in the disk $|z|<k, k\geq 0$ except for t-fold zeros at origin then for $0\leq r\leq R\geq k$,

$$Max_{|z|=R}|P'(z)| \leq \frac{(n-t)R^{\mu-1}}{R^{\mu}+k^{\mu}}.\frac{(R^{\mu}+k^{\mu})^{\frac{n-t}{\mu}}}{(r^{\mu}+k^{\mu})^{\frac{n-t}{\mu}}}\left\{\frac{R^t}{r^t}Max_{|z|=r}|P(z)|\right.$$

$$\left.-\frac{R^t}{k^t}Min_{|z|=k}|P(z)|\right\}+\frac{t}{R}Max_{|z|=R}|P(z)|. \qquad (21)$$

The result is best possible and equality holds for the polynomial $P(z) = (z^{\mu}+k^{\mu})^{\frac{n}{\mu}}$ where n is a multiple of μ.

Remark 4. Taking $t = 0$ in Theorem 1.3 we get Theorem B.
If we take $\mu = t+1, r = 1$ in Theorem 1.3, we get the following result.

Corollary 1.5. If $P(z) = z^t \sum_{j=t+1}^{n} a_j z^j)$ is a polynomial of degree n, having no zero in the disk $|z| < k, k \geq 1$ except for t-fold zeros at origin then for $1 \leq R \leq k$,

$$Max_{|z|=R}|P'(z)| \leq \frac{(n-t)R^t}{R^{t+1}+k^{t+1}}.\frac{(R^{t+1}+k^{t+1})^{\frac{n-t}{t+1}}}{(1+k^{t+1})^{\frac{n-t}{t+1}}}\left\{R^t Max_{|z|=1}|P(z)|\right.$$

$$\left.-\frac{R^t}{k^t}Min_{|z|=k}|P(z)|\right\}+\frac{t}{R}Max_{|z|=R}|P(z)|. \qquad (22)$$

The result is sharp and equality holds for $P(z) = (z^{\mu}+k^{\mu})^{\frac{n}{\mu}}, \mu = t+1$
If we take $R = k = 1$ in corollary 1.5 we get the following generalization of a result due to Aziz and Dawood [3]

$$Max_{|z|=1}|P'(z)| \leq \frac{n+t}{2}Max_{|z|=1}|P(z)| - \frac{n-t}{2}Min_{|z|=1}|P(z)|.$$

The result is best possible for $P(z) = (z+k)^n$.
If we take $R = k = 1$ in Theorem 1.3, we get the following generalization of a result due to Aziz and shah [4, Cor. 6].

Corollary 1.6. If $P(z) = z^t(a_t + \sum_{j=\mu}^{n} a_j z^{j-t}), t+1 \leq \mu \leq n$, is a polynomial of degree n, having no zero in the disk $|z| < 1$, except for t-fold zeros at origin then for $0 < r \leq 1$,

$$Max_{|z|=1}|P'(z)| \leq \frac{n-t}{2}.(\frac{2}{1+r^{\mu}})^{\frac{n-t}{\mu}}\frac{1}{r^t}\left\{Max_{|z|=r}|P(z)|\right.$$

$$\left.-Min_{|z|=1}|P(z)|\right\}+tMax_{|z|=1}|P(z)|.$$

The result is best possible and equality holds for the polynomial $P(z) = (z^{\mu}+1)^{\frac{n}{\mu}}$ where n is a multiple of μ.

Lemmas

For the proofs of these Theorems we need the following Lemmas. The following Lemma is due to Aziz and Shah [4].

Lemma 1. If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ is a polynomial of degree n having no zeros in the disk $|z| \leq k, k \geq 1$ then for every $R > 1$ and $|z| = 1$

$$|P(Rz) - P(z)| \leq \frac{1}{k^{\mu+1}} \left\{ \frac{1 + \frac{R^\mu - 1}{R^n - 1} |\frac{a_\mu}{a_0}| k^{\mu+1}}{\frac{R^\mu - 1}{R^n - 1} |\frac{a_\mu}{a_0}| k^{\mu-1} + 1} \right\} |Q(Rz) - Q(z)| \quad (23)$$

Lemma 2. If $P(z)$ is a polynomial of degree n Then for every $R > 1$

$$|P(Rz) - P(z)| + |Q(Rz) - Q(z)| \leq (R^\mu - 1) Max_{|z|=1} |P(z)| \quad (24)$$

The above Lemma due to Aziz, [2](see also [10])

Lemma 3. If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ is a polynomial of degree n having no zeros in the disk $|z| < k, k \geq 1$ then

$$Max_{|z|=1} |P'(z)| \leq \frac{1}{1 + k^\mu} \left\{ Max_{|z|=1} |P(z)| - Min_{|z|=k} |P(z)| \right\} \quad (25)$$

The Lemma was proved by Dewan and pukhta, [15].

Lemma 4. Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ be a polynomial of degree n such that

$$M(P, r) = Max_{|z|=r} |P(z)| \quad \text{and} \quad m(P, r) = Min_{|z|=r} |P(z)|.$$

If $P(z)$ has no zeros in $|z| < k, k > 0$ then for $0 \leq r \leq R \leq k$.

$$M(P, r) \geq \left(\frac{r^\mu + k^\mu}{R^\mu + k^\mu} \right)^{\frac{n}{\mu}} M(P, R) + \left\{ 1 - \left(\frac{r^\mu + k^\mu}{R^\mu + k^\mu} \right)^{\frac{n}{\mu}} \right\} m(P, k). \quad (26)$$

The result is sharp and equality holds for the polynomial $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is multiple of μ.
Lemma 4 is due to Aziz and Shah [4].

Proofs of Theorems

Proof of Theorem 1.1. Since $P(z) = z^t(a_t + \sum_{j=\mu}^{n} a_j z^{j-t}) = z^t H(z), t+1 \leq \mu \leq n$ does not vanish in $|z| < k, k \geq 1$ except for t-fold zeros at origin.Applying Lemma 1 to the polynomial $H(z)$ of degree $n-t$, we get,

$$\frac{k^{\mu+1}\left(\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|k^{\mu-1}+1\right)}{1+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|k^{\mu+1}}|H(Rz)-H(z)| \leq |G(Rz)-G(z)| \qquad (27)$$

Where,

$$G(z) = z^{n-t}\overline{H(\frac{1}{\bar{z}})}.$$

Inequality (27) with the help of Lemma 2 implies that

$$\left\{1+\frac{k^{\mu+1}\left(\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|k^{\mu-1}\right)+1}{1+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|k^{\mu+1}}\right\}|H(Rz)-H(z)|$$

$$\leq |H(Rz)-H(z)| + |G(Rz)-G(z)|$$

$$\leq (R^{n-t}-1)Max_{|z|=1}|H(z)|.$$

This gives

$$|H(Rz)-H(z)| \leq (R^{n-t}-1)\left(\frac{1+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|k^{\mu+1}}{1+k^{\mu+1}+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})}\right) Max_{|z|=1}|H(z)|$$

or

$$|R^t z^t H(Rz)-R^t z^t H(z)| \leq (R^n-R^t)\left(\frac{1+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|k^{\mu+1}}{1+k^{\mu+1}+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})}\right) Max_{|z|=1}|H(z)|.$$

This gives,

$$|P(Rz)-R^t P(z)| \leq (R^n-R^t)\left(\frac{1+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|k^{\mu+1}}{1+k^{\mu+1}+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})}\right) Max_{|z|=1}|P(z)|$$

which is inequality (10) and this proves Theorem 1.1 completely.

Proof of Theorem 1.2. From inequality (10) it follows that

$$|P(Rz)-P(z)+P(z)-R^t P(z)| \leq (R^n-R^t)\left(\frac{1+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|k^{\mu+1}}{1+k^{\mu+1}+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})}\right) Max_{|z|=1}|P(z)|$$

$$|P(Rz)-P(z)| \leq \left((R^t-1)+(R^n-R^t)\left\{\frac{1+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|k^{\mu+1}}{1+k^{\mu+1}+\frac{R^{\mu}-1}{R^{n-t}-1}|\frac{a_\mu}{a_t}|(k^{2\mu}+k^{\mu+1})}\right\}\right) Max_{|z|=1}|P(z)|$$

which is inequality (11) and hence Theorem 1.2 is proved.

Proof of Theorem 1.3. By hypothesis $P(z)=z^t(a_t+\sum_{j=\mu}^{n} a_j z^{j-t})=z^t H(z)$, $t+1\leq\mu\leq n$ does not vanish in $|z|<k, k\geq 1$ except for t-fold zeros at origin, therefore the polynomial $F(z)=H(Rz)$ has no zeros in $|z|<\frac{k}{R}, \frac{k}{R}\geq 1$ Applying Lemma 3 to the polynomial $F(z)$ we get,
we get

$$|F'(z)| \leq \frac{n-t}{1+\frac{k^\mu}{R^\mu}}\left(Max_{|z|=1}|F(z)|-Min_{|z|=\frac{k}{R}}|F(z)|\right).$$

which gives,

$$Max_{|z|=R}|H'(z)| \leq \frac{(n-t)R^{\mu-1}}{R^\mu+k^\mu}\left\{Max_{|z|=R}|H(z)|-Min_{|z|=k}|H(z)|.\right\} \qquad (28)$$

Now if $0 \leq r \leq R \leq k$,then by Lemma 4 we have,

$$Max_{|z|=R}|H(z)| \leq \left(\frac{r^{\mu}+k^{\mu}}{R^{\mu}+k^{\mu}}\right)^{\frac{n-t}{\mu}} Max_{|z|=r}|H(z)|$$

$$+\left\{1-\left(\frac{r^{\mu}+k^{\mu}}{R^{\mu}+k^{\mu}}\right)^{\frac{n-t}{\mu}}\right\}Min_{|z|=k}|H(z)| \qquad (29)$$

From (28) and (29) it follows that

$$Max_{|z|=R}|H'(z)| \leq \frac{(n-t)R^{\mu-1}}{R^{\mu}+k^{\mu}}\left\{\left(\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n-t}{\mu}} Max_{|z|=r}|H(z)| - \left(\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n-t}{\mu}} Min_{|z|=k}|H(z)|.\right\}$$

$$=\frac{(n-t)R^{\mu-1}}{R^{\mu}+k^{\mu}}\left(\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n-t}{\mu}}\left\{Max_{|z|=r}|H(z)|-Min_{|z|=k}|H(z)|\right\}$$

since

$$P'(z)=z^{t}H'(z)+tz^{t-1}H(z)$$

$$Max_{|z|=R}|P'(z)|=Max_{|z|=R}|z^{t}H'(z)+tz^{t-1}H(z)|$$

$$\leq R^{t}Max_{|z|=R}|H'(z)|+tR^{t-1}Max_{|z|=R}|H(z)|$$
$$=R^{t}\left(\frac{(n-t)R^{\mu-1}}{R^{\mu}+k^{\mu}}\left\{\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right\}^{\frac{n-t}{\mu}}\left\{Max_{|z|=r}|H(z)|-Min_{|z|=k}|H(z)|\right\}\right) + \frac{t}{R}R^{t}Max_{|z|=R}|H(z)|$$

$$= \frac{(n-t)R^{\mu-1}}{R^{\mu}+k^{\mu}}\left\{\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right\}^{\frac{n-t}{\mu}}\left(\frac{R^t}{r^t}Max_{|z|=r}|r^tH(z)| - R^tMin_{|z|=k}|H(z)|\right) + \frac{t}{R}Max_{|z|=R}|P(z)|$$

$$= \frac{(n-t)R^{\mu-1}}{R^{\mu}+k^{\mu}}\left\{\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right\}^{\frac{n-t}{\mu}}\left(\frac{R^t}{r^t}Max_{|z|=r}|P(z)| - \frac{R^t}{r^t}Min_{|z|=k}|P(z)|\right) + \frac{t}{R}Max_{|z|=R}|P(z)|$$

which is equivalent to inequality(21) and this completes the proof of Theorem 1.3.

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