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On the regions containing zeros and zero free regions of a Polynomial

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Abstract

If $P(z) = \sum_{j=0}^{n} a_j z^j, a_j \geq a_{j-1}, a_0 > 0, j = 1, 2, \cdots, n$ is a polynomial of degree n, then according to a classical result of Eneström-Kakeya, all the zeros of $P(z)$ lie in $|z| \leq 1$. Joyal (et al) [9] extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non-negative.In this paper, I will prove some extensions and generalizations of this result by relaxing the hypothesis.

Key words: *Polynomial, Zeros*, Eneström-Kakeya *Theorem*
Mathematics Subject Classification: 30C10, 30C15.

1.INTRODUCTION AND STATEMENTS OF RESULTS

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, then concerning the distribution of zeros of $P(z)$, Eneström and Kakeya [10, 11] proved the following interesting result.

Theorem A. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0, \tag{1}$$

then $P(z)$ has all its zeros in $|z| \leq 1$.

In the literature [1-11], there exist several extensions and generalizations of this Theorem. Joyal *et al* [9] extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non-negative. In fact they proved the following result.

Theorem B. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

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then $P(z)$ has all its zeros in the disk

$$|z| \leq \frac{1}{|a_n|}\left(|a_n| - a_0 + |a_0|\right).$$

In this paper, I will prove some generalizations and extensions of Theorem B and hence of the Theorem A i,e Eneström-Kakeya Theorem. In this direction I first present the following interesting result in which we relax the hypothesis and hence is a generalization of Theorem B. In fact, I prove the following:

Theorem 1. *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + a_{p-1}z^{p-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n satisfying $a_p \geq a_{p-1} \geq \cdots \geq a_1 \geq a_0, p = 0, 1, \cdots, n$* and $M_p = \sum\limits_{j=p+1}^{n} |a_j - a_{j-1}|$, *then all the zeros of $P(z)$ lie in the disk*

$$|z| \leq \frac{M_p + a_p - a_o + |a_0|}{|a_n|}$$

Remark 1. For $p = n$, Theorem 1 reduces to Theorem B.
Applying Theorem 1 to the polynomial $P(tz)$, we get the following result:

Corollary 1. *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + a_{p-1}z^{p-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n such that for any $t > 0$,*

$$t^p a_p \geq t^{p-1}a_{p-1} \geq \cdots \geq t a_1 \geq a_0, p = 0, 1, \cdots, n$$

then all the zeros of $P(z)$ lie in the disk

$$|z| \leq \sum_{j=p+1}^{n} \frac{|ta_j - a_{j-1}|}{t^{n-j+1}|a_n|} + \frac{t^p a_p - a_0 + |a_0|}{t^n |a_n|}$$

The following result follows from Corollary 1 by taking $p = 0$.

Corollary 2. *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n, then for any $t > 0$ all the zeros of $P(z)$ lie in the disk*

$$|z| \leq \sum_{j=0}^{n} \frac{|ta_j - a_{j-1}|}{t^{n-j+1}|a_n|}.$$

We also prove the following result which gives the lower bound for the moduli of zeros of a polynomial. In other words it provides the zero free region for polynomials.

Theorem 2. *Let* $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + a_{p-1}z^{p-1} + \cdots + a_1 z + a_0$ *be a polynomial of degree* n *satisfying* $a_p \geq a_{p-1} \geq \cdots \geq a_1 \geq a_0, p = 0, 1, \cdots, n$ and $M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}|$, *then* $P(z)$ *does not vanish in*

$$|z| < Min\left\{1, \frac{|a_0|}{|a_n| + M_p + a_p - a_o}\right\}$$

For $p = n$, Theorem 2 reduces to the following result:
Corollary 3. *Let* $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ *be a polynomial of degree* n *satisfying,*$a_n \geq a_{p-1} \geq \cdots \geq a_1 \geq a_0$ *then* $P(z)$ *does not vanish in*

$$|z| < \frac{|a_0|}{|a_n| + a_n - a_0}.$$

The bound is attained by the polynomial $P(z) = z^n + z^{n-1} + \cdots + z + 1$.

Next we prove the following more general result which is also a generalization of Theorem B.

Theorem 3. *Let* $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + \cdots + a_1 z + a_0$ *be a polynomial of degree* n *satisfying*

$$a_p \geq a_{p-1} \geq \cdots \geq a_0, \qquad 0 \leq p \leq n$$

and

$$\underset{|z|=1}{Max}\left|\sum_{j=p+1}^{n} (a_j - a_{j-1})z^{n-j}\right| \leq M,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \leq max\left(1, \frac{|a_0| - a_0 + a_p + M}{|a_n|}\right).$$

Remark 2. Let $\underset{|z|=1}{Max}\left|\sum_{j=p+1}^{n}(a_j - a_{j-1})z^j\right|$ is attained at $z = e^{i\alpha}$, then,

$$\begin{aligned} M &= \left|\sum_{j=p+1}^{n}(a_j - a_{j-1})e^{i\alpha}\right| \\ &\leq \sum_{j=p+1}^{n}\left|a_j - a_{j-1}\right| = M_p, \;\; 0 \leq p \leq n, \end{aligned}$$

where M_p is defined as in Theorem 1. Thus

$$M \leq M_p, \;\; 0 \leq p \leq n.$$

From this, we conclude that Theorem 3 is a refinement of Theorem 1.

The following result is an immediate consequence of the Theorem 3.

Corollary 4. *Let* $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ *be a polynomial of degree* n*, then all the zeros of* $P(z)$ *lie in*

$$|z| \leq \frac{M}{|a_n|},$$

where

$$M = \underset{|z|=1}{Max}\left|\sum_{j=0}^{n}(a_j - a_{j-1})z^{n-j}\right|.$$

1 Proofs of the Theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)\big(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\big) \\ &= a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 - a_n z^{n+1} \\ &- a_{n-1}z^n - \cdots - a_0 z \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} \\ &+ \cdots + (a_1 - a_0)z + a_0. \end{aligned}$$

This gives

$$|F(z)| \geq \left|a_n z^{n+1}\right| - \Big\{ \left|a_n - a_{n-1}\right||z|^n + \left|a_{n-1} - a_{n-2}\right||z|^{n-1}$$

$$+ \cdots + \left|a_{p+1} - a_p\right||z|^{p+1} + \cdots + \left|a_1 - a_0\right||z| + |a_0| \Big\}$$

$$= |z|^n \Big\{ |a_n||z| - \Big(|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|}$$

$$+ \cdots + \frac{|a_{p+1} - a_p|}{|z|^{n-p-1}} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Big) \Big\}.$$

Now, let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1, 0 \leq j \leq n$, then we have

$$|F(z)| > |z|^n \Big\{ |a_n||z| - \Big(|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}|$$

$$+ \cdots + |a_{p+1} - a_p| + \cdots + |a_1 - a_0| + |a_0| \Big) \Big\}$$

$$= |z|^n \Big\{ |a_n||z| - \Big(|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots$$

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$$+ |a_{p+1} - a_p| + a_p - a_{p-1} + \cdots + a_1 - a_0 + |a_0|\Big)\Big\}$$

$$= |z|^n\Big\{|a_n||z| - \Big(\sum_{j=p+1}^{n} |a_j - a_{j-1}| + a_p - a_0 + |a_0|\Big)\Big\}$$

$$= |z|^n\Big\{|a_n||z| - \Big(M_p + a_p - a_0 + |a_0|\Big)\Big\}$$

$$> 0, \quad if|z||a_n| > \Big(M_p + a_p - a_0 + |a_0|\Big),$$

$$i,e \ if \quad |z| > \frac{\Big(M_p + a_p - a_0 + |a_0|\Big)}{|a_n|}$$

where $M_p = \sum_{j=p+1}^{n} \big|a_j - a_{j-1}\big|$. Thus all the zeros of $F(z)$ whose modulus is greater than 1 lie in the disk

$$|z| \leq \frac{1}{|a_n|}\Big(M_p + a_p - a_0 + |a_0|\Big).$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality and all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $P(z)$ lie in the disk

$$|z| \leq \frac{1}{|a_n|}\Big(M_p + a_p - a_0 + |a_0|\Big).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Consider the reciprocal polynomial

$$R(z) = z^n P(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_p z^{n-p} + \cdots + a_n.$$

Let

$$\begin{aligned} S(z) &= (1-z)R(z) \\ &= -a_0 z^{n+1} + (a_0 - a_1) z^n + \cdots \\ &\quad + (a_p - a_{p+1}) z^{n-p} + \cdots + (a_{n-1} - a_n) z + a_n. \end{aligned}$$

This gives

$$\begin{aligned} \big|S(z)\big| &\geq |a_0||z|^{n+1} - \Big\{ |a_0 - a_1||z|^n + \cdots + |a_p - a_{p+1}||z|^{n-p} \\ &\qquad + \cdots + |a_{n-1} - a_n||z| + |a_n| \Big\} \\ &= |z|^n \Big\{ |a_0||z| - \Big(|a_0 - a_1| + \cdots + \frac{|a_p - a_{p+1}|}{|z|^p} \\ &\qquad + \cdots + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \Big) \Big\}. \end{aligned}$$

Now, let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1, 0 \leq j \leq n$, then we have

$$
\begin{aligned}
\big|S(z)\big| \geq |z|^n \Bigg\{ |a_0||z| - \Bigg(&|a_0 - a_1| + \cdots + |a_p - a_{p+1}| + \cdots \\
&+ |a_{n-1} - a_n| + |a_n| \Bigg) \Bigg\} \\
= |z|^n \Bigg\{ |a_0||z| - \Bigg(&|a_1 - a_0| + \cdots + |a_{p+1} - a_p| + |a_p - a_{p-1}| \\
&+ \cdots + |a_n - a_{n-1}| + |a_n| \Bigg) \Bigg\} \\
= |z|^n \Bigg\{ |a_0||z| - \Bigg(&a_p - |a_0| + |a_n| + \sum_{j=p+1}^{n} |a_j - a_{j-1}| \Bigg) \Bigg\} \\
= |z|^n \Bigg\{ |a_0||z| - \Bigg(&a_p - |a_0| + |a_n| + M_p \Bigg) \Bigg\} \\
&> 0, \ \text{if} \\
&|z| > \frac{1}{|a_0|} \Bigg\{ a_p - |a_0| + |a_n| + M_p \Bigg\},
\end{aligned}
$$

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where $M_p = \sum_{j=p+1}^{n} \left|a_j - a_{j-1}\right|$. Thus all the zeros of $S(z)$ whose modulus is greater than 1 lie in

$$|z| \leq \frac{1}{|a_0|}\left\{a_p - |a_0| + |a_n| + M_p\right\}.$$

Hence all the zeros of $S(z)$ and hence of $R(z)$ lie in

$$|z| \leq Max\left\{1, \frac{1}{|a_0|}\left(a_p - |a_0| + |a_n| + M_p\right)\right\}.$$

Therefore all the zeros of $P(z)$ lie in

$$|z| \geq Min\left\{1, \frac{|a_0|}{a_p - |a_0| + |a_n| + M_p}\right\}.$$

$$|z| \geq Min\left\{1, \frac{|a_0|}{a_p - |a_0| + |a_n| + M_p}\right\}.$$

Thus the polynomial $P(z)$ does not vanish in

$$|z| < Min\left(1, \frac{|a_0|}{a_p - |a_0| + |a_n| + M_p}\right).$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Consider the polynomial

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$$
\begin{aligned}
&F(z)\\
&= (1-z)P(z)\\
&= (1-z)\big(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\big)\\
&= a_n z^n + \cdots + a_1 z + a_0 - a_n z^{n+1} - a_{n-1}z^n - \cdots - a_0 z\\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_{p+1} - a_p)z^{p+1}\\
&+ (a_p - a_{p-1})z^p + \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0\\
&= R(z) - a_n z^{n+1},
\end{aligned}
$$

where

$$
\begin{aligned}
R(z) = (a_n - a_{n-1})z^n + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p\\
+ \cdots + (a_1 - a_0)z + a_0.
\end{aligned}
$$

Let

$$
\begin{aligned}
R^*(z) &= z^n R(1/z)\\
&= a_0 z^n + (a_1 - a_0)z^{n-1} + \cdots + (a_p - a_{p-1})z^{n-p}\\
&+ (a_{p+1} - a_p)z^{n-p-1} + \cdots + (a_n - a_{n-1}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|R^*(z)| \le &\left|a_0 z^n + (a_1 - a_0)z^{n-1} \cdots + (a_p - a_{p-1})z^{n-p}\right|\\
&+ \left|(a_{p+1} - a_p)z^{n-p-1} + \cdots + (a_n - a_{n-1})\right|\\
&\le |a_0||z|^n + |(a_1 - a_0)|\,|z|^{n-1} + \cdots + |(a_p - a_{p-1})|\,|z|^{n-p} + \Big|\sum_{j=p+1}^{n} (a_j - a_{j-1})z^{n-j}\Big|\\
&\le |a_0| + a_p - a_0 + M, \quad \text{for } |z| = 1,
\end{aligned}
$$

where M is defined as in the statement of the Theorem. Hence by maximum modulus principle, it follows that

$$|R^*(z)| \leq |a_0| + a_p - a_0 + M, \quad \text{for } |z| \leq 1.$$

Therefore

$$|R(z)| \leq |z|^n(|a_0| + a_p - a_0 + M), \quad \text{for } |z| \geq 1.$$

This gives for $|z| > 1$,

$$\begin{aligned}
|F(z)| &\geq \left|a_n z^{n+1}\right| - \left|R(z)\right| \\
&\geq \left|a_n z^{n+1}\right| - z^n(|a_0| + a_p - a_0 + M) \\
&= \left|a_n\right||z|^n\left\{|z| - \frac{|a_0| + a_p - a_0 + M}{\left|a_n\right|}\right\} \\
&> 0, \text{ if} \\
|z| &> \frac{|a_0| + a_p - a_0 + M}{|a_n|}.
\end{aligned}$$

Thus all zeros of $F(z)$ whose modulus is greater than 1 lie in the disk

$$|z| \leq \frac{|a_0| + a_p - a_0 + M}{\left|a_n\right|}.$$

Therefore all zeros of $F(z)$ lie in the disk

$$|z| \leq Max\left\{1, \frac{|a_0| + a_p - a_0 + M}{\left|a_n\right|}\right\}.$$

But all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $P(z)$ lie in the disk

$$|z| \leq Max\left\{1, \frac{|a_0| + a_p - a_0 + M}{\left|a_n\right|}\right\}.$$

This completes the proof of Theorem 3.

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