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Fixed Point Theorem of Partial Fuzzy Metric Space

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ABSTRACT

Fuzzy metric space have been defined using fuzzy numbers by many researchers. Recently Zun-Quan Xia and Fang Fang Guo [1] *have introduced fuzzy metric space using fuzzy scalars which is similar to the 'classical metric space'. In this paper we have defined 'partial fuzzy metric' space using fuzzy scalars. We have also proved the fixed point theorem for 'partial fuzzy metric space.'*

Key Words: *Fuzzy metric space, fuzzy partial metric space, fixed point theorem , fuzzy numbers, fuzzy scalars*

I. INTRODUCTION

Many authors have developed and used the theory of fuzzy sets (introduced by Zadeh) in many different fields. A lot of researchers have studied 'fuzzy metric spaces' and fixed point theorems related to it. [1-4].

Among the various approaches to define 'fuzzy metric space'. Some of the researchers used fuzzy numbers & others have used fuzzy points on real space R to measure the distance between fuzzy points [5]. In this paper, we have defined a 'partial fuzzy metric space' using fuzzy points . We have also established the proof of fixed point theorem for 'partial fuzzy metric space'.

II. PRELIMINARIES

In order to find the main result, we have taken some basic and relevant definitions from the literature which is already existing.

Definition 2.1:[1] Fuzzy Points**:** A fuzzy set in X is said to be fuzzy point if and only if it is of the form:

$X_\lambda(y) = \begin{cases} \lambda , x = y \\ 0 , x \neq y \end{cases}$ where X is non-empty & $\lambda \in [0, 1]$.

We denote set of fuzzy points as (x, λ) .

Definition 2.2 :[2] Fuzzy Scalars**:** Let (x, λ) and (y,γ) be fuzzy points. Then they are fuzzy scalars if they satisfy the following conditions:

(a) (a, λ) $\geq (b,\gamma), if$ a $> b$ or (a, λ) $= (b,\gamma)$.

(b) (a, λ) $\nless (b,\gamma)$, if a $\geq$ b.

(c) (a, λ) is non-negative if a $\geq$ 0.

Set of all fuzzy points on X is denoted as $P_F(X)$. Whenever X = R set of fuzzy points will be set of fuzzy scalars denoted as $S_F(R)$and $S_F^+(R)$ is the set of all non negative fuzzy scalars.

Definition 2.3: Consider X as a set which is non-empty and also consider a mapping $p_F: P_F(X) \times P_F(X) \rightarrow S_F^+(R)$. Then we say that $(P_F(X), p_F)$ is a 'partial fuzzy metric space' if for any (x,λ), $(y,\gamma), (z,\rho) \subset P_F(X)$, p_F satisfies the below mentioned conditions:

i. $x = y \leq p_F\big((x,\lambda),(x,\lambda)\big) = p_F\big((x,\lambda),(y,\gamma)\big) = p_F\big((y,\gamma),(y,\gamma)\big)$

ii. $p_F\big((x,\lambda),(x,\lambda)\big) \leq p_F\big((x,\lambda)(y,\gamma)\big)$

iii. $p_F\big((x,\lambda),(y,\gamma)\big) = p_F\big((y,\gamma),(x,\lambda)\big)$

iv. $p_F((x,\lambda),(y,\gamma) \leq p\big((x,\lambda),(z,\rho)\big) + p_F\big((z,\rho),(y,\gamma)\big) - (p_F(z,\rho),(z,\rho))$

A fuzzy partial metric space is a pair (X, p_F) such that X is a non-empty set & p_F is a partial metric on X.

Definition2.4: Any sequence $\{(x_n,\lambda_n)\}$ in a fuzzy partial metric space $(P_F(x), p_F)$ converges to a particular point xϵ X if and only if $\underset{n\to\infty}{lt}\ p_F\big((x_n,\lambda_n),(x,\lambda)\big) = p_F\big((x,\lambda),(x,\lambda)\big)$.

Definition 2.5: Any sequence $\{(x_n,\lambda_n)\}$ in a fuzzy partial metric space $(P_F(x), p_F)$ is called a cauchy sequence if there exists a finite $\underset{n,m\to\infty}{lt} p_F(x_m,\lambda_m),(x_n,\lambda_n)$ for all $m, n\epsilon N$.

Definition 2.6: A partial fuzzy metric space is complete if every cauchy sequence $\{(x_n,\lambda_n)\}$ converges in $P_F(x)$ which means that $\underset{n,m\to\infty}{lt} p_F(x_m,\lambda_m),(x_n,\lambda_n) = p_F\big((x,\lambda),(x,\lambda)\big)$.

Lemma 2.1: Let $(P_F(x), p_F)$ be a fuzzy partial metric space then it is complete if and only if the the metric space (X, p) is compelete. Also, $\lim_{n\to\infty} p_F((x_n, \lambda_n), (x, \lambda)) = 0$ if and only if $\lim_{n,m\to\infty} p_F(x_m, \lambda_m), (x_n, \lambda_n) = p_F((x_n, \lambda_n), (x, \lambda)) = p_F((x, \lambda), (x, \lambda))$.

III. MAIN RESULTS

Theorem 3.1- Let there be a 'fuzzy partial metric space' P_F on $P_F(x)$, and $(P_F(x), P_F)$ be a complete 'fuzzy partial metric space.' Let T: $P_F(x) \to P_F(x)$ be a continuous and non-decreasing mapping. If $\exists$ $x_0 \epsilon$ X such that $(x_0, \lambda_0) \leq T(x_0, \lambda_0)$ then $\exists$ $x \epsilon$ X such that Tx = x . Moreover, $p_F((x, \lambda), (x, \lambda)) = 0$.

Proof: Let $T(x_0, \lambda_0) = (x_0, \lambda_0)$ then there is nothing to prove.

Let $(x_0, \lambda_0) \neq T(x_0, \lambda_0)$. Also let $(x_n, \lambda_n) = T(x_{n-1}, \lambda_{n-1})$.

We need to show that $(x_{n_0}, \lambda_{n_0}) = (x_{n_{0+1}}, \lambda_{n_{0+1}})$ as it will prove that (x_{n_0}, λ_{n_0}) is a fixed point of T.

$\therefore$ Let $(x_{n+1}, \lambda_{n+1}) \neq (x_n, \lambda_n)$.

As, $(x_{n_0}, \lambda_{n_0}) \leq T(x_0, \lambda_0)$, and T is non-decreasing for every n $\in$ N.

$\therefore$ $(x_0, \lambda_0) \leq (x_1, \lambda_1) \leq (x_1, \lambda_1) \leq (x_2, \lambda_2) \leq \ldots\ldots \leq (x_n, \lambda_n) \leq (x_{n+1}, \lambda_{n+1}) \ldots\ldots\ldots$

As $(x_{n-1}, \lambda_{n-1}) \leq (x_n, \lambda_n)$.

$\therefore$ $p_F((x_{n+1}, \lambda_{n+1}), (x_n, \lambda_n)) = p_F(T(x_n, \lambda_n), T(x_{n-1}, \lambda_{n-1}))$.

$\leq p_F((x_n, \lambda_n), (x_{n-1}, \lambda_{n-1}))$ - $\phi(p_F((x_n, \lambda_n), (x_{n-1}, \lambda_{n-1}))$.

$< p_F(x_n, \lambda_n), (x_{n-1}, \lambda_{n-1})$

$\therefore$ p_{f_n} ={ $p_F((x_{n+1}, \lambda_{n+1}), (x_n, \lambda_n)$} is a non-negative and decreasing sequence.

$\therefore$ It has a limit p. Let p> 0 then $\exists$ $n_0 \in$ N such that $\phi(p_{f_n}) \geq \phi(p) > 0$ for all n $> n_0$

and $p_{f_n} \leq p_{f_{n-1}}$ - $\phi(p_{f_{n-1}}) \leq p_{f_{n-1}}$ - $\phi(p)$.

Taking limit as n→ ∞ we have:

$p \leq$ p - $\phi(p) \leq$ p which is a contradiction. ∴ $p_F = 0$.

Now we need to show that (x_n, λ_n) is a cauchy sequence.

Since, $\underset{n\to\infty}{lt}\ p_F(x_n, \lambda_n), (x_{n-1}, \lambda_{n-1}) = 0$

We can easily show that $\underset{n\to\infty}{lt}\ p_F(x_{n+1}, \lambda_{n+1}), (x_n, \lambda_n) = 0$. See [6].

Hence, (x_n, λ_n) is a cauchy sequence.

Also, as $\underset{n\to\infty}{lt}\ p_F((x_{n+1}, \lambda_{n+1}), T(x, \lambda)) = p_F((T(x, \lambda), T(x, \lambda))$.

And, $p_F((x, \lambda), T(x, \lambda)) \leq p_F((x, \lambda), (x_{n+1}, \lambda_{n+1})) + p_F((x_{n+1}, \lambda_{n+1}), T(x, \lambda))$

$$- p_F((x_{n+1}, \lambda_{n+1}), (x_{n+1}, \lambda_{n+1}))$$

$$\leq p_F((x_{n+1}, \lambda_{n+1}), (x, \lambda)) + p_F((x_{n+1}, \lambda_{n+1}), T(x, \lambda))$$

Letting n→ ∞ we get :

$p_F((x, \lambda), T(x, \lambda)) \leq p_F((T(x, \lambda), T(x, \lambda))$

$$\leq p_F((x, \lambda), (x, \lambda)) - \phi(p_F((x, \lambda), (x, \lambda)) = 0.$$

∴ $p_F((x, \lambda), T(x, \lambda)) = 0$. Hence, $T(x, \lambda) = (x, \lambda)$.

Lemma 3.1 - [6] For every $(x, \lambda), (y, \gamma) \in p_F(X)$ ∃ an upper bound or a lower bound. That is, for all x, y ∈ X ∃ z ∈ X which is comparable to (x, λ) *and* (y, γ).

Theorem 3.2 - We now prove the uniqueness of T using the lemma 3.1.

Proof: Let ∃ (z, ρ) *and* $(y, \gamma) \in p_F(X)$ which are two different fixed points of T.

∴ $p_F((z, \rho), (y, \gamma)) > 0$. Now,

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(i) If (z,ρ) and (y,γ) are comparable then

$T^n(z,\rho) = (z,\rho)$ and $T^n(y,\gamma) = (y,\gamma)$ are comparable for n = 0,1,2…

$$\therefore \quad p_F((z,\rho),(y,\gamma)) \leq p_F(T^n(z,\rho), T^n(y,\gamma))$$

$$\leq p_F(T^{n-1}(z,\rho), T^{n-1}(y,\gamma))$$

$$- \phi(T^{n-1}(z,\rho), T^{n-1}(y,\gamma))$$

$$= p_F((z,\rho),(y,\gamma)) - \phi\big(p_F(z,\rho),(y,\gamma)\big)$$

$$< p_F((z,\rho),(y,\gamma)).$$

which is a contradiction.

(ii) If (z,ρ) and (y,γ) are comparable then $\exists$ x $\in X$ comparable to z & y.

Since, T is increasing therefore $T^n(x)$ is comparable to $T^n(z,\rho) = (z,\rho)$

and $T^n(y,\gamma) = (y,\gamma)$ for n = 0,1,2……

$$\therefore p_F(z, T^n x) = pT^n((z,\rho), T^n(x,\lambda)$$

$$\leq p_F\big(T^{n-1}(z,\rho), T^{n-1}(x,\lambda)\big) - \phi p_F(T^{n-1}(z,\rho), T^{n-1}(x,\lambda))$$

$$= p_F((z,\rho), T^{n-1}(x,\lambda)) - \phi p_F((z,\rho), T^{n-1}(x,\lambda))$$

$$< p_F\big((z,\rho), T^{n-1}(x,\lambda)\big)$$

Also, $\lim_{n\to\infty} P((y,\gamma), T^n(x,\lambda)) = 0$

$$p_F\big((z,\rho),(y,\gamma)\big) \leq p((z,\rho), T^n(x,\lambda))$$

taking limit $n \to \infty$ gives $p\big((z,\rho),(y,\gamma)\big) = 0$ which is a contradiction.

$$\therefore \quad p_F\big((z,\rho),(y,\gamma)\big) > 0.$$

Hence, proved.

IV. CONCLUSION

In this paper we have generalized the concept of a fuzzy metric space to a partial fuzzy metric space. We have also introduced contraction mapping & fixed point theorem for the same. These spaces have many applications in fuzzy optimization and pattern recognition. This paper also opens scope to further generalize this concept to other spaces like banach space or a fuzzy topological space.

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