SUBCLASSES OF ANALYTIC FUNCTIONS RELATED TO SIGMOID FUNCTION

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ABSTRACT

In this paper, the authors investigate the initial coefficient bounds for some new subclasses of analytic functions related to Sigmoid function. Also the relevant connections to Fekete-Szegö inequality and Hankel determinant for these classes are briefly discussed. Our results serve as a new generalization in this direction.

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I INTRODUCTION AND PRELIMINARIES

The theory of special functions is significantly important to scientists and engineers. Though not with any specific definition but its applications extend to physics, computer etc. Recently, the theory of special functions has been overshadowed by other fields like real analysis, functional analysis, algebra, topology and differential equations.

There are various special functions but we shall concern with one of the activation function known as sigmoid function or simple logistic function. Activation function is an information process that is inspired by the biological nervous system such as brain processes information. It comprises of large number of highly interconnected processing element (neurons) working together to solve a specific task. The function works in similar way the brain does, it learns by examples and cannot be programmed to solve a specific task.

The sigmoid function of the form

$$h(z)=\frac{1}{1+e^{-z}} \tag{1.1}$$

is differentiable and has the following properties:

- It outputs real numbers between 0 and 1.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is a one-to-one function.
- It increases monotonically.

The four properties above shows that sigmoid function is very useful in geometric function theory.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.2}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$.

Let U be the class of bounded functions

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \tag{1.3}$$

which are regular in the unit disc and satisfying the conditions

$$w(0) = 0 \text{ and } |w(z)| < 1 \text{ in } E.$$

For functions f and g analytic in *E*, we say that f is subordinate to g, denoted by $f \prec g$, if there exists a Schwarz function $w(z) \in U$, $w(z)$ analytic in *E* with $w(0) = 0$ and $|w(z)| < 1$ in *E*, such that $f(z) = g(w(z))$. It follows from Schwarz lemma that $f(z) \prec g(z)(z \in E) \Rightarrow f(0) = g(0)$ and $f(E) \subset g(E)$ (see detail in [1]).

Fekete and Szegö in 1933 gave the sharp bound for the functional $|a_3 - \mu a_2^2|$ for the functions in the class *S* of univalent functions when μ is real. The determination of the sharp bounds for the functional $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegö problem and this has been investigated by several authors for different subclasses of univalent functions.

In 1976, Noonan and Thomas [2] stated the *qth* Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. Easily, one can observe that the Fekete and Szegö functional is $H_2(1)$.

For $q = 2$ and $n = 2$, $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$ is the second Hankel determinant.

A function $f(z)\in A$ is said to be in the class $f(z)\in H(\lambda)$ if

$$\mathrm{Re}\left[(1-\lambda)f'(z)+\lambda\frac{(zf'(z))'}{f'(z)}\right]>0.$$

This class was introduced by Al-Amiri and Reade [3]. In particular $H(0)\equiv R,$ the class of functions whose derivative has a positive real part and studied by Macgregor [4].

A function $f(z)\in A$ is said to be in the class $f(z)\in R(\lambda)$ if it satisfies

$$\mathrm{Re}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\right]>0.$$

The class $R(\lambda)$ was studied by Murugusundramurthi and Magesh [5] and in particular $R(1)\equiv R.$

Motivated by above defined classes, we introduce the following subclasses of analytic functions of complex order related to sigmoid functions.

DEFINITION 1.1 For $b\in C.$ Let the class $H_\lambda(b,\Phi_{m,n})(\lambda\geq 0)$ denote the subclass of A consisting of functions of the form (1.2) and satisfying the following condition

$$1+\frac{1}{b}\left[(1-\lambda)f'(z)+\lambda\frac{(zf'(z))'}{f'(z)}-1\right]>0,$$

for $0\leq\lambda\leq 1$ and $\Phi_{m,n}(z)$ is a simple logistic sigmoid activation function.

DEFINITION 1.2 For $b\in C.$ Let the class $R_\lambda(b,\Phi_{m,n})(\lambda\geq 0)$ denote the subclass of A consisting of functions of the form (1.2) and satisfying the following condition

$$1+\frac{1}{b}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)-1\right]>0,$$

for $0\leq\lambda\leq 1$ and $\Phi_{m,n}(z)$ is a simple logistic sigmoid activation function.

Recently, various authors as Abiodun [6], Murugusundramoorthy and Janani [7], Olatunji et al.[8] and Olatunji [9] have studied sigmoid function for different classes of analytic and univalent functions.

In the present work, we obtained few coefficient bounds for the classes $H_\lambda(b,\Phi_{m,n})$ and $R_\lambda(b,\Phi_{m,n})$ and the relevant connection with Fekete-Szegö theorems and Hankel determinant.

To prove our result we shall make use of the following lemmas:

LEMMA 1.1 [10] If a function $p\in P$ is given by

$$p(z)=1+p_1z+p_2z^2+\ldots(z\in E),$$

then $|p_k| \le 2, k \in N$ where *P* is the family of all functions analytic in *E* for which $p(0)=1$ and $\mathrm{Re}(p(z)) > 0 (z \in E)$.

LEMMA 1.2 [11] Let h be the sigmoid function defined in (1.1) and

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m,$$

then $\Phi(z) \in P, |z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

LEMMA 1.3 [11] Let

$$\Phi_{m,n}(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m,$$

then $|\Phi_{m,n}(z)| < 2.$

LEMMA 1.4 [11] If

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n = \frac{(-1)^{n+1}}{2n!}$, then $|c_n| \le 2, n \in N$ and the result is sharp for each *n*.

II. INITIAL COEFFICIENTS

THEOREM 2.1 If $f(z) \in A$ of the form (1.2) is belonging to $H_\lambda(b, \Phi_{m,n})$, then

$$|a_2| \le \frac{|b|}{4}, \tag{2.1}$$

$$|a_3| \le \frac{|b|^2 \lambda}{12(1+\lambda)} \tag{2.2}$$

and

$$|a_4| \le \frac{|b[\lambda(6b^2\lambda - 3b^2 - 1) - 1]|}{96(1+\lambda)(1+2\lambda)}. \tag{2.3}$$

Proof. As $f(z) \in H_\lambda(b, \Phi_{m,n})$, therefore

$$1 + \frac{1}{b}\left[(1-\lambda) f'(z) + \lambda \frac{(zf'(z))'}{f'(z)} - 1 \right] = \Phi_{m,n}(z) \tag{2.4}$$

where
$$\Phi_{m,n}(z)=1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{1}{64}z^6+\frac{779}{20160}z^7-\ldots \quad (2.5)$$

Using (2.5), (2.4) can be expanded as

$$2a_2z+\left[3(1+\lambda)a_3-4\lambda a_2^2\right]z^2+\left[4(1+2\lambda)a_4-18\lambda a_2a_3+8\lambda a_2^3\right]z^3+\ldots=b\left[\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\ldots\right]. \quad (2.6)$$

Equating the coefficients of z, z^2 and z^3 in (2.6), we obtain

$$a_2=\frac{b}{4}, \quad (2.7)$$

$$a_3=\frac{b^2\lambda}{12(1+\lambda)} \quad (2.8)$$

and

$$a_4=\frac{b\left[\lambda\left(6b^2\lambda-3b^2-1\right)-1\right]}{96(1+\lambda)(1+2\lambda)}. \quad (2.9)$$

Results (2.1), (2.2) and (2.3) can be easily obtained from (2.7), (2.8) and (2.9) respectively.

For $b=1$ in Theorem 2.1, the following result is obvious:

COROLLARY 2.1 If $f(z)\in H_\lambda(1,\Phi_{m,n})$, then

$$|a_2|\le\frac{1}{4}, \qquad |a_3|\le\frac{\lambda}{12(1+\lambda)} \qquad \text{and} \qquad |a_4|\le\frac{|\{\lambda(6\lambda-4)-1\}|}{96(1+\lambda)(1+2\lambda)}.$$

THEOREM 2.2 If $f(z)\in A$ of the form (1.2) is belonging to $R_\lambda(b,\Phi_{m,n})$, then

$$|a_2|\le\frac{|b|}{2(1+\lambda)}, \quad (2.10)$$

$$|a_3|\le 0 \quad (2.11)$$

and

$$|a_4|\le\frac{|b|}{24(1+3\lambda)}. \quad (2.12)$$

Proof. Since $f(z)\in R_\lambda(b,\Phi_{m,n})$, therefore

$$1+\frac{1}{b}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)-1\right]=\Phi_{m,n}(z) \quad (2.13)$$

where $$\Phi_{m,n}(z)=1+\frac{1}{2}z-\frac{1}{24}z^{3}+\frac{1}{240}z^{5}-\frac{1}{64}z^{6}+\frac{779}{20160}z^{7}-\ldots \qquad (2.14)$$

Using (2.14) in (2.13), it yields

$$(1+\lambda)a_2 z+(1+2\lambda)a_3 z^2+(1+3\lambda)a_4 z^3+\ldots=b\left[\frac{1}{2}z-\frac{1}{24}z^3+\ldots\right] \qquad (2.15)$$

Equating the coefficients of z, z^2 and z^3 in (2.15), we obtain

$$a_2=\frac{b}{2(1+\lambda)}, \qquad (2.16)$$

$$a_3=0 \qquad (2.17)$$

and

$$a_4=-\frac{b}{24(1+3\lambda)}. \qquad (2.18)$$

Results (2.10), (2.11) and (2.12) can be easily obtained from (2.16),(2.17) and (2.18) respectively.

For $b=1$ in Theorem 2.2, the following result is obvious:

COROLLARY 2.2 If $f(z)\in R_\lambda(1,\Phi_{m,n})$, then

$$|a_2|\le\frac{1}{2(1+\lambda)}, \qquad |a_3|\le 0 \qquad \text{and} \qquad |a_4|\le\frac{1}{24(1+3\lambda)}.$$

III. FEKETE-SZEGÖ INEQUALITY

THEOREM 3.1 If $f(z)\in A$ of the form (1.2) is belonging to $H_\lambda(b,\Phi_{m,n})$, then

$$\left|a_3-\mu a_2^2\right|\le\frac{|b|^2}{4}\left|\frac{\lambda}{3(1+\lambda)}-\mu\frac{1}{4}\right|. \qquad (3.1)$$

Proof. From (2.7) and (2.8), we have

$$a_3-\mu a_2^2=\frac{b^2}{4}\left[\frac{\lambda}{3(1+\lambda)}-\mu\frac{1}{4}\right]. \qquad (3.2)$$

Hence (3.1) can be easily obtained from (3.2).

THEOREM 3.2 If $f(z)\in A$ of the form (1.2) is belonging to $R_\lambda(b,\Phi_{m,n})$, then

$$\left|a_3-\mu a_2^2\right|\le\frac{|b^2\mu|}{4(1+\lambda)^2}. \qquad (3.3)$$

Proof. Using (2.16) and (2.17), we have

$$a_3-\mu a_2^2=\frac{b^2\mu}{4(1+\lambda)^2}. \tag{3.4}$$

Hence (3.3) can be easily obtained from (3.4).

IV. SECOND HANKEL DETERMINANT

THEOREM 4.1 If $f(z)\in A$ of the form (1.2) is belonging to $H_\lambda(b,\Phi_{m,n})$, then

$$\left|a_2a_4-\mu a_3^2\right|\le\frac{|b|^2}{48(1+\lambda)}\left|\frac{\left[\lambda\left(6b^2\lambda-3b^2-1\right)-1\right]}{8(1+2\lambda)}-\mu\frac{b^2\lambda^2}{3(1+\lambda)^2}\right|. \tag{4.1}$$

Proof. From (2.7), (2.8) and (2.9), we have

$$a_2a_4-\mu a_3^2=\frac{b^2}{48(1+\lambda)}\left[\frac{\left[\lambda\left(6b^2\lambda-3b^2-1\right)-1\right]}{8(1+2\lambda)}-\mu\frac{b^2\lambda^2}{3(1+\lambda)^2}\right]. \tag{4.2}$$

Hence (4.1) can be easily obtained from (4.2).

THEOREM 4.2 If $f(z)\in A$ of the form (1.2) is belonging to $R_\lambda(b,\Phi_{m,n})$, then

$$\left|a_2a_4-\mu a_3^2\right|\le\frac{|b|^2}{48(1+\lambda)(1+3\lambda)}. \tag{4.3}$$

Proof. From (2.16), (2.17) and (2.18), we have

$$a_2a_4-\mu a_3^2=\frac{b^2}{48(1+\lambda)(1+3\lambda)}. \tag{4.4}$$

Hence (4.3) can be easily obtained from (4.4).

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