International Journal of Advance Research in Science and Engineering
Volume No.07, Special Issue No.08, March 2018
www.ijarse.com
IJARSE
ISSN: 2319-8354

COEFFICIENT INEQUALITY FOR A SUBCLASS OF ANALYTIC FUNCTIONS USING SUBORDINATION METHOD WITH EXTREMAL FUNCTION

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ABSTRACT

Here we introduced a new advance subclass $P(\delta,\gamma)$ of class $P(\delta)$ of analytic functions by using principle of subordination also obtained its sharp upper bound and Fekete Szego inequality for function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1.$

Keywords: *Analytic functions, Bounded functions, Fekete Szego Inequality, Univalent function.*

I INTRODUCTION

The class of function of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic in the unit disk $E = \{z : |z| < 1\}$ are denoted by class A

The class of function of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$ are denoted by class S.

In (1916), Bieberbach [4, 5] proved a very useful result for the function of class S i.e. if $f(z) \in S$ then $|a_2| \le 2.$ after this he stated that this result is also true for all values of $n.$ If $f(z) \in S$ then $|a_n| \le n \; \forall n \in \mathrm{N}$

In (1923), Lowner [2] proved the above result for third coefficient that $|a_3| \le 3$. And it was natural to find out some relation between a_3 and a_2^2 for the class S this famous relation was obtained by Fekete and Szego [6] with the help of Lowner's method.

Let $f(z) \in S$, then $\left|a_3 - \mu a_2^2\right| \le \begin{cases} 3-4\mu & \text{if} \quad \mu \le 0 \\ 1+\exp\left(\dfrac{-2\mu}{1-\mu}\right) & \text{if} \quad 0 \le \mu \le 1 \\ 4\mu-3 & \text{if} \quad \mu \ge 1 \end{cases}$

This inequality is very much helpful in obtaining estimates of higher coefficients for some subclasses of S (See Chhichra [1], Babalola [3]).

Now we define some subclasses of S

A function $f(z) \in A$ is convex function if it is univalent in E and $f(E)$ is a convex domain. We denote the class of convex functions by K.

For a function $f(z) \in A$ if there exists a function

$$g(z) = \sum_{k=1}^{\infty} b_k z^k, \operatorname{Re}(b_1) > 0$$

which is univalent and starlike in E such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in E$$

This class of close to convex function was introduced by Kaplan in 1952 and

we denote the class of close to convex functions by C.

Analytic bounded functions: Class of analytic bounded function is of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, w(0) = 0, |w(z)| \le 1.$$

It is known that $|c_1| \le 1, |c_2| \le 1 - |c_1|^2$.

Here we have a class as

$S^*(f(f(z))) = \left\{ f(z) \in A; \frac{zf'(f(z))f'(z)}{f(f(z))} \prec \frac{1+z}{1-z}, z \in E \right\}$ and the important subclasses of this function are

- $S^*(f(f(z)),\delta) = \left\{ f(z) \in A; \frac{zf'(f(z))f'(z)}{f(f(z))} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, z \in E \right\}$
- $S^*(f(f(z)),A,B) = \left\{ f(z) \in A; \frac{zf'(f(z))f'(z)}{f(f(z))} \prec \left(\frac{1+Az}{1+Bz}\right), z \in E \right\}$
- $S^*(f(f(z)),\delta,A,B) = \left\{ f(z) \in A; \frac{zf'(f(z))f'(z)}{f(f(z))} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}, z \in E \right\}$ Here the symbol $\prec$ stands for subordination.

II MAIN RESULTS

Theorem 1 : Let $f(z) \in S^*(f(f(z)))$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1}{2} - \mu \; ; if \;\; \mu \le 0; \\ \frac{1}{2} \; ; if \;\; 0 \le \mu \le 1; \\ \mu - \frac{1}{2} \; ; if \;\; \mu \ge 1 \end{cases}$$

Proof: By definition of $f(z) \in S^*(f(f(z)))$, we have $\frac{zf'(f(z))f'(z)}{f(f(z))} \prec \frac{1+w(z)}{1-w(z)}$

While expanding the above series , we get

$$1 + 4a_2 z + (6a_3 + 6a_2^2)z^2 + \ldots = (1 + (2a_2 + 2c_1)z + (4a_2c_1 + 2c_1^2 + 2c_2 + 2a_3 + 2a_2^2)z^2 + \ldots)$$

Comparing coefficients

$a_2 = c_1$ and

$$a_3 = \frac{c_2 + c_1^2}{2}$$

By using above values, we obtain

IJARSE
ISSN: 2319-8354

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{2}\left|c_2\right|+\left|\frac{1}{2}-\mu\right|\left|c_1\right|^2$$

by using def. of bounded analytic function we redefine the above equation as

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{2}+\left[\left|\frac{1}{2}-\mu\right|-\frac{1}{2}\right]\left|c_1\right|^2 \qquad \ldots *$$

CASE 1: when $\mu\le\frac{1}{2}$ equation * can be rewritten as

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{2}-\mu\left|c_1\right|^2 \qquad \ldots **$$

SUBCASE 1(a): when $\mu\le 0$ the equation ** redefined as

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{2}-\mu \qquad \ldots(A)$$

SUBCASE 1(b): when $\mu\ge 0$ the equation ** redefined as

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{2} \qquad \ldots(B)$$

CASE 2: when $\mu\ge\frac{1}{2}$ equation * can be rewritten as

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{2}+(\mu-1)\left|c_1\right|^2 \qquad \ldots ***$$

SUBCASE 2(a): when $\mu\le 1$ the equation *** redefined as

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{2} \qquad \ldots(C)$$

SUBCASE 2(b): when $\mu\ge 1$ the equation *** redefined as

$$\left|a_3-\mu a_2^2\right|\le\mu-\frac{1}{2} \qquad \ldots(D)$$

By combining the above (A,B,C,D) inequalities we get our result

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{2}-\mu\,;\, if \;\; \mu \le 0; \\ \frac{1}{2}\,;\, if \;\; 0 \le \mu \le 1; \\ \mu - \frac{1}{2}\,;\, if \;\; \mu \ge 1 \end{cases}$$

Thus the theorem is proved.

The extremal function for the 1st and 3rd inequality is $z\left(1-z+\frac{1}{2}z^2\right)^{-1}$

The extremal function for 2nd inequality is $\frac{z}{(1-2z)^{\frac{1}{4}}}$

Theorem 2 : Let $f(z) \in S^*\left(f\left(f(z)\right), A, B\right)$then

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{B(B-A)}{4}-\mu\frac{(A-B)^2}{4}; & if \;\; \mu \le \frac{1+B}{B-A}; \\ \frac{A-B}{4}; & if \;\; \frac{1+B}{B-A} \le \mu \le \frac{1-B}{A-B}; \\ \frac{B(A-B)}{4}\mu\frac{(A-B)^2}{4}; & if \;\; \mu \ge \frac{1-B}{A-B} \end{cases}$$

Proof: By definition we have $\frac{zf'\left(f(z)\right)f'(z)}{f\left(f(z)\right)} \prec \frac{1+Aw(z)}{1+Bw(z)}$

By expanding the series

$$\begin{aligned} &1+4a_2z+\left(6a_3+6a_2^2\right)z^2+\cdots \\ &=1+\left((A-B)c_1+2a_2\right)z+\left(2a_3+2a_2^2\right)+(A-B)c_2+\left(B^2-AB\right)c_1^2+2a_2(A-B)c_1 \end{aligned}$$

Comparing coefficients of ()

$$a_2=\frac{(A-B)c_1}{2}$$
$$a_3=\frac{(A-B)c_2+\left(B^2-AB\right)c_1^2}{4}$$

By using above values, we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{A-B}{4} + \frac{1}{4}\left[\left|B^2 - AB - \mu(A-B)^2\right| - (A-B)\right]\left|c_1\right|^2 \qquad \ldots *$$

CASE 1: when $\mu \le \dfrac{B^2 - AB}{(A-B)^2}$ equation * can be rewritten as

$$\left|a_3 - \mu a_2^2\right| \le \frac{A-B}{4} - \frac{1}{4}\left[(B+1)(A-B) + \mu(A-B)^2\right]\left|c_1\right|^2 \qquad \ldots **$$

SUBCASE 1(a): when $\mu \le \dfrac{1+B}{B-A}$ the equation ** redefined as

$$\left|a_3 - \mu a_2^2\right| \le \frac{B(B-A)}{4} - \mu\frac{(A-B)^2}{4} \qquad \ldots (A)$$

SUBCASE 1(b): when $\mu \ge \dfrac{1+B}{B-A}$ the equation ** redefined as

$$\left|a_3 - \mu a_2^2\right| \le \frac{A-B}{4} \qquad \ldots (B)$$

CASE 2: when $\mu \ge \dfrac{B^2 - AB}{(A-B)^2}$ equation * can be rewritten as

$$\left|a_3 - \mu a_2^2\right| \le \frac{A-B}{4} - \frac{1}{4}\left[(1-B)(A-B) - \mu(A-B)^2\right]\left|c_1\right|^2 \qquad \ldots ***$$

SUBCASE 2(a): when $\mu \le \dfrac{1-B}{A-B}$ the equation *** redefined as

$$\left|a_3 - \mu a_2^2\right| \le \frac{A-B}{4} \qquad \ldots (C)$$

SUBCASE 2(b): when $\mu \ge \dfrac{1-B}{A-B}$ the equation *** redefined as

$$\left|a_3 - \mu a_2^2\right| \le \frac{B(A-B)}{4} + \mu\frac{(A-B)^2}{4} \qquad \ldots (D)$$

By combining the above (A,B,C,D) inequalities we get our result

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{B(B-A)}{4}-\mu\frac{(A-B)^2}{4}; & if\ \ \mu\le\frac{1+B}{B-A};\\ \frac{A-B}{4}; & if\ \ \frac{1+B}{B-A}\le\mu\le\frac{1-B}{A-B};\\ \frac{B(A-B)}{4}\mu\frac{(A-B)^2}{4}; & if\ \ \mu\ge\frac{1-B}{A-B}\end{cases}$$

Thus the theorem is proved.

The extremal function for the 1st and 3rd inequality is $z\left(1+\frac{A+3B}{2}z\right)^{\frac{A-B}{A+3B}}$

The extremal function for 2nd inequality is $\frac{z}{\left(1+2Bz^2\right)^{\frac{B-A1}{8}}}$

Corollary 1: Putting $A=1, B=-1$ in above theorem, we get

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{1}{2}-\mu\,; if\ \ \mu\le 0;\\ \frac{1}{2}\,; if\ \ 0\le\mu\le 1;\\ \mu-\frac{1}{2}\,; if\ \ \mu\ge 1\end{cases}$$

Thus this is the result of theorem 1

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