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Continued Finite Fractions and Euclid's Algorithm

Parvinder Singh
Principal, S.G.G. S. Khalsa College, Mahilpur. (Hoshiarpur).

ABSTRACT

A "general" continued fraction representation of a real number x is one of the form

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots \cdots \cdots + \cfrac{b_n}{a_N}}}}$$

Where a_0 , a_1, ... and b_1, b_2 … ..are integers. In this article we define convergents of a finite continued fraction and continued fractions with positive quotients and discuss fraction algorithm and Euclid's algorithm.

INTRODUCTION:

Define a function $f(\text{n}) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cdots \cdots + \cfrac{1}{a_N}}}}$ ……………….(a)

Consisting of N + 1 variables $a_0, a_1, \dots \dots, a_N$ as a finite continued fraction. As the representation (a) is cumbersome, we shall usually write it as $[a_0, a_1, \dots \dots, a_n]$ and we call $a_0, a_1, \dots.., a_n$ the partial quotients or simply the quotients of the finite continued fraction. As above we see that $[a_0] = \frac{a_0}{1}$, $[a_0, a_1] = \frac{a_0 a_1 + 1}{a_1}$,$[a_0, a_1, a_2] = \frac{a_2 a_1 a_0 + a_2 + a_0}{a_2 a_1 + 1}$ ……. Therefore $[a_0, a_1] = a_0 + \frac{1}{a_1}$ and

Similarly $[a_0, a_1, \dots \dots, a_{n-1}, a_n]$= $[a_0, a_1, \dots.., a_{n-2}, a_{n-1} + \frac{1}{a_n}]$…………..(1.1)

i.e. $[a_0, a_1, \dots \dots, a_n] = a_0 + \frac{1}{[a_0, a_1, \dots \dots, a_n]} = [a_0, [a_0, a_1, \dots \dots, a_n]]$, for $1 \leq n \leq$ N

Moreover $[a_0, a_1, \dots \dots, a_n] = [a_0, a_1, \dots \dots, a_{m-1,} [a_m, a_{m+1}, \dots \dots, a_n]]$ for $1 \leq n \leq$ N.

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Definition: The quantity $[a_0, a_1, \dots \dots, a_n]$ for ($1 \leq n \leq$ N) is called nth convergent to $[a_0, a_1, \dots \dots, a_N]$. Also it is easy to find the convergents by means of the following theorem.

Theorem 1.1: Let p_n and q_n be defined as under $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ ($2 \leq n \leq$ N) and

$q_1 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ ($2 \leq n \leq$ N) then $[a_0, a_1, \dots \dots, a_n] = \frac{p_n}{q_n}$.

Proof: For n=1 and n =1 theorem is obviously true.

Let suppose that result holds for n $\leq m$, where m < N. Then

$[a_0, a_1, \dots \dots, a_{m-1,} a_m] = \frac{p_m}{q_m} = \frac{a_m \, p_{m-1+} \, p_{m-2}}{a_m \, q_{m-1} + q_{m-2}}$, and p_{m-1}, p_{m-2}, q_{m-1}, q_{m-2} depend only upon $a_0, a_1, \dots \dots, a_{m-1}$.

Hence using (1.1) we get $[a_0, a_1, \dots \dots, a_{m-1,} a_m, a_{m+1}] = [a_0, a_1, \dots \dots, a_{m-1,} a_m + \frac{1}{a_{m+1}}] = \frac{\left(a_m + \frac{1}{a_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(a_m + \frac{1}{a_{m+1}}\right) q_{m-1} + q_{m-2}} = \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}} = \frac{a_{m+1} p_m + p_{m-1}}{a_{m+1} q_m + q_{m-1}} = \frac{p_{m+1}}{q_{m+1}}$

Hence by induction the theorem is proved.

Note: From $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n \, p_{n-1} + p_{n-2}$ ($2 \leq n \leq$ N) and

$q_1 = 1$, $q_1 = a_1$, $q_n = a_n \, q_{n-1} + q_{n-2}$ ($2 \leq n \leq$ N) it follows that

$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$

Also $p_n q_{n-1}$ - $p_{n-1} q_n = (a_n \, p_{n-1} + p_{n-2}) \, q_{n-1} - p_{n-1}(a_n \, q_{n-1} + q_{n-2})$

$= - (p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$.

Repeating the argument with n-1, n-2,……,2 in place of n, we get

$p_n q_{n-1}$ - $p_{n-1} q_n = (-1)^{n-1} (p_1 q_0 - p_0 q_1) = (-1)^{n-1}$.

Also $p_n q_{n-2}$ - $p_{n-2} q_n = (a_n \, p_{n-1} + p_{n-2}) \, q_{n-2}$ -$p_{n-2}(a_n \, q_{n-1} + q_{n-2})$

$= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = (-1)^{n-1} \, a_n$.

Remark: The functions p_n and q_n satisfies the following.

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$p_n q_{n-1}$- $p_{n-1}q_n$ = $(-1)^{n-1}$ or $\frac{p_n}{q_n}$ - $\frac{p_{n-1}}{q_{n-1}}$ = $\frac{(-1)^{n-1}}{q_{n-1}\, q_n}$

Also they satisfy $p_n q_{n-2}$- $p_{n-2}q_n$ = $(-1)^{n-1}a_n$ or $\frac{p_n}{q_n}$ - $\frac{p_{n-2}}{q_{n-2}}$ = $\frac{(-1)^{n-1}a_n}{q_{n-2}\, q_n}$.

Definition: Now we assign numerical values to the quotients a_n so to the fraction $a_0 + \cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots\ldots\ldots+\cfrac{1}{a_N}}}}$ and to its convergents.

Now suppose that $a_1 > 0$,…….., $a_N > 0$, a_0 may be negative , in this case the continued fraction is said to be simple. Write $x_n = \frac{p_n}{q_n}$, x = x_N so that the value of the continued fraction is x_N or x. Then

$[a_0, a_1, \ldots\ldots, a_N] = [a_0, a_1, \ldots\ldots, a_{n-1,}[a_n, a_{n+1}, \ldots\ldots, a_N]]$

$= \frac{[a_n, a_{n+1},\ldots\ldots, a_N]p_{n-1} + p_{n-2}}{[a_n, a_{n+1},\ldots\ldots, a_N]q_{n-1} + q_{n-2}}$ for $2 \leq n \leq$ N.

Note: As every q_n is positive then from $\frac{p_n}{q_n}$ - $\frac{p_{n-2}}{q_{n-2}}$ = $\frac{(-1)^{n-1}a_n}{q_{n-2}\, q_n}$ and $a_1 >$ 0,…….., $a_N > 0$, $x_n - x_{n-2}$ has the sign of $(-1)^n$. Which proves that the even convergents x_{2n} increase strictly with n, while the odd convergents x_{2n+1} decrease strictly.

Also from $\frac{p_n}{q_n}$ - $\frac{p_{n-1}}{q_{n-1}}$ = $\frac{(-1)^{n-1}}{q_{n-1}\, q_n}$, $x_n - x_{n-1}$ has the sign of $(-1)^{n-1}$

so that $x_{2m+1} > x_{2m}$ contrary if we assume that $x_{2m+1} \leq x_{2\mu}$ for some m, μ.If m $< \mu$ then from above $x_{2m+1} < x_{2m}$, and if m $< \mu$ then $x_{2\mu+1} < x_{2\mu}$which is a contradiction. Hence we say that every odd convergent is greater than any even convergent.

Definition: If all a_n are integers then the continued fraction is called Simple Fraction. If p_n and q_n are integers and q_n is positive then

$[a_0, a_1, \dots\dots, a_N]$ = $\frac{p_N}{q_N}$ = x, we say that the number x (which is necessarily rational) is represented by the continued fraction.

Theorem 1.2: $q_n \geq n$, with inequality when n > 3.

Proof: In the first place, q_0 = 1, q_1 = a_1 ≥ 1. If n ≥ 2 then

$q_n = a_n q_{n-1} + q_{n-2} \geq q_{n-1}+1$ so that $q_n > q_{n-1}$ and $q_n \geq n$. If n > 3, then

$q_n \geq q_{n-1} + q_{n-2} > q_{n-1}+1 \geq$ n, and so $q_n > n$.

Definition: Any simple continued fraction $[a_0, a_1, \dots\dots, a_N]$ represents a rational number x = x_N

Theorem 1.3: If x is representable by a simple continued fraction with an odd (even) number of convergents, it is also representable by one with an even (odd) number.

Proof: If $a_n \geq 2$ then $[a_0, a_1, \dots\dots, a_n]$ = $[a_0, a_1, \dots\dots, a_n - 1, 1]$ while, if a_n = 1, $[a_0, a_1, \dots\dots, a_{n-1}, 1]$ = $[a_0, a_1, \dots\dots, a_{n-2}, a_n + 1]$

For example [2,2,3] = [2,2,2,1] this choice of alternative representations is often useful. We call a_n' = $[a_n, a_{n+1}, \dots\dots, a_N]$ ($0 \leq n \leq$ N) the nth complete quotient of the continued fraction $[a_0, a_1, \dots\dots, a_N]$. Thus x = a_0', x = $\frac{a_1' a_0 + 1}{a_1'}$ and

x = $\frac{a_n' p_{n-1} + p_{n-2}}{a_n' q_{n-1} + q_{n-2}}$, $(2 \leq n \leq N)$ …………….(b)

Theorem 1.4: a_n = $[a_n']$, the integral part of a_n' except that a_{N-1} = $[a_{N-1}] - 1$ when a_N = 1.

Proof: If N = 0, then $a_0 = a_0' = [a_0']$. If N > 0 then $a_n' = a_n + \frac{1}{a_{n+1}'}$ $(0 \leq n \leq N-1)$.

Now a_{n+1}' > 1 $(0 \leq n \leq N-1)$ except that a_{n+1}' = 1 when n = N -1 and a_N = 1.

Hence $a_n < a_n^{'} < a_n + 1$ $(0 \le n \le$ N-1) and $a_n = [a_n^{'}]$ for $(0 \le n \le$ N-1) except in the case specified. And in any case $a_N = a_N^{'} = [a_N^{'}]$.

Theorem 1.5: If two simple continued fractions $[a_0, a_1, \dots\dots, a_N]$ and $[b_0, b_1, \dots\dots, b_M]$ have the same value x, and $b_M > 1$, then M = N and the fractions are identical.

Proof: When we say that the two continued fractions are identical we mean that they are formed by the same sequence of partial quotients.

By the above theorem a_0 = [x] = b_0. Let us suppose that the first n partial quotients in the continued fractions are identical and that $a_n^{'}$ *and* $b_n^{'}$ are the nth complete quotients. Then x = $[a_0, a_1, \dots\dots, a_{n-1}, a_n^{'}] = [a_0, a_1, \dots\dots, a_{n-1}, b_n^{'}]$.

If n = 1 then $a_0 + \frac{1}{a_1^{'}} = a_0 + \frac{1}{b_1^{'}}$, $a_1^{'} = b_1^{'}$, and therefore by above theorem $a_1 = b_1$.

If n > 1, then by $\frac{a_n^{'} p_{n-1} + p_{n-2}}{a_n^{'} q_{n-1} + q_{n-2}} = \frac{b_n^{'} p_{n-1} + p_{n-2}}{b_n^{'} q_{n-1} + q_{n-2}}$,

$(a_n^{'} - b_n^{'})(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}) = 0$. But $p_{n-1}q_{n-2} - p_{n-2}q_{n-1} = (-1)^n$ then as $p_n q_{n-1} - p_{n-1}q_n = (-1)^{n-1}$ and so $a_n^{'} = b_n^{'}$, it follows from the above theorem that $a_n = b_n$.

Suppose now for example, that N ≤ M. Then our argument shows that $a_n = b_n$ for N ≤ M. If M > N then $\frac{p_N}{q_N} = [a_0, a_1, \dots\dots, a_N] = [a_0, a_1, \dots\dots, a_N, b_{N+1}, \dots\dots, b_M]$

$= \frac{b_{N+1}^{'} p_N + p_{N-1}}{b_{N+1}^{'} q_N + q_{N-1}}$, Hence by (b) $p_N q_{N-1} - p_{N-1} q_N = 0$ which is false. Hence M = N and the fractions are identical.

Continued fraction algorithm and Euclid's algorithm:

Let x be any real number, and let a_0 = [x]. Then x = $a_0 + \xi_0$, $0 \le \xi_0 < 1$.

If $\xi_0 \neq 0$, we can write $\frac{1}{\xi_0} = a_1^{'}$, $[a_n^{'}] = a_1$, $a_1^{'} = a_1 + \xi_1$, $0 \le \xi_1 < 1$.

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If $\xi_1 \neq 0$, we can write $\frac{1}{\xi_1} = a_2' = a_2 + \xi_2, 0 \leq \xi_2 < 1$, and so on

Also $a_n' = \frac{1}{\xi_{n-1}} > 1$, and so $a_n \geq 1$, for n≥ 1. Thus x = $[a_0, a_1'] = [a_0, a_1 + \frac{1}{a_2'}] = [a_0, a_1, a_2'] = [a_0, a_1, a_2, a_3'] = \ldots\ldots$ where $a_0, a_1, a_2, \ldots\ldots$ are integers and $a_1 > 0, a_2 > 0, \ldots\ldots\ldots$

The system of equations x = $a_0 + \xi_0$, $(0 \leq \xi_0 < 1)$,

$$\frac{1}{\xi_0} = a_1' = a_1 + \xi_1, \ (0 \leq \xi_1 < 1),$$

$$\frac{1}{\xi_1} = a_2' = a_2 + \xi_2, \ (0 \leq \xi_2 < 1),$$

………….. is known as the continued fraction algorithm. The algorithm continues so long as $\xi_n \neq 0$. If we eventually reach a value of n, say N, for which $\xi_N = 0$, the algorithm terminates and x = $[a_0, a_1, \ldots\ldots, a_N]$.

In this case x is represented by a simple continued fraction, and is rational. The number a_n' are the complete quotients of the continued fraction.

Theorem 1.6: Any rational number can be represented by a finite simple continued fraction.

Proof: If x is an integer, then $\xi_0 = 0$ and x = a_0. If x is not integral, then x = $\frac{h}{k}$, where h and k are integers and k > 1. Since $\frac{h}{k} = a_0 + \xi_0$, h = a_0k + ξ_0k, a_0 is the quotient, and $k_1 = \xi_0$k the remainder, when h is divided by k.

If $\xi_0 \neq 0$ then $a_1' = \frac{1}{\xi_0} = \frac{k}{k_1}$ and $\frac{k}{k_1} = a_1 + \xi_1$, k = $a_1 k_1 + \xi_1 k_1$; thus a_1 is the quotient, and $k_2 = \xi_1 k_1$ the remainder, when k is divided by k_1.Thus we obtain a series of equations h = a_0k + k_1 , k = $a_1 k_1 + k_2$, $k_1 = a_2 k_2 + k_3$,…………….

Continuing so long as $\xi_n \neq 0$, or what is the same thing, so long as $k_{n+1} \neq 0$.

The non-negative integers k, $k_1, k_2, \dots \dots$.... form a strictly decreasing sequence, and so k_{n+1} = 0 for some N. It follows that ξ_N = 0 for some N, and the continued fraction algorithm terminates. This proves the theorem.

Remark: The system of equations

h = a_0k + k_1 , (0 < $k_1 < k$),

k = $a_1 k_1 + k_2$, (0 < $k_2 < k_1$),

………………………………..

$k_{N-2} = a_{N-1}k_{N-1} + k_N$, (0 < $k_N < k_{N-1}$),

$k_{N-1} = a_N k_N$ is known as Euclid's algorithm.

Difference between the fraction and its convergents:

Suppose N > 1 and n > 0 then by x = $\frac{a_n' p_{n-1} + p_{n-2}}{a_n' q_{n-1} + q_{n-2}}$, (1 ≤ n ≤ N-1) and so

x - $\frac{p_n}{q_n}$ = - $\frac{p_n q_{n-1} - p_{n-1} q_n}{q_n (a_{n+1}' q_n + q_{n-1})}$ = $\frac{(-1)^n}{q_n (a_{n+1}' q_n + q_{n-1})}$, Also x - $\frac{p_0}{q_0}$ = x - a_0 = $\frac{1}{a_1'}$.

If we write $q_1' = a_1'$, $q_n' = a_n' q_{n-1} + q_{n-2}$, (1 ≤ n ≤ N-1)

(So in particular $q_N' = q_N$), we have the following theorem.

Theorem 1.7: If 1 ≤ n ≤ N-1, then x - $\frac{p_n}{q_n}$ = $\frac{(-1)^n}{q_n \, q_{n+1}'}$

Proof: $a_{n+1} < a_{n+1}' < a_{n+1}$ +1 for n ≤ $N-2$,

by the equation $a_n < a_n' < a_n + 1$ (0 ≤ n ≤ N-1), except that $a_{N-1}' = a_{N-1}$ +1 when a_N =1. Hence if we ignore this exceptional case for the moment, we have

$q_1 = a_1 < a_1' + 1 \leq q_2$ and $q_{n+1}' = a_{n+1}' q_n + q_{n-1} > a_{n+1} q_n + q_{n-1} = q_{n+1}$

$q_{n+1}' < a_{n+1} q_n + q_{n-1} + q_n = q_{n+1} + q_n \leq a_{n+2} q_{n+1} + q_n = q_{n+2}$,

for $1 \leq n \leq$ N-2. It follows that $\frac{1}{q_{n+2}} < |p_n - q_n \mathrm{x}| < \frac{1}{q_{n+1}}$ $(n \leq$ N-2) while $|p_{N-1} - q_{N-1}\mathrm{x}| = \frac{1}{q_N}$, $p_N - q_N \mathrm{x} = 0$ in the exceptional case $q'_{n+1} < a_{n+1}q_n + q_{n-1} + q_n = q_{n+1} + q_n \leq a_{n+2}q_{n+1} + q_n = q_{n+2}$ must be replaced by
q'_{N-1} = ($|a_{N-1}$ + 1) $q_{N-2} + q_{N-3} = q_{N-1} + q_{N-2} = q_N$ and the first inequality. In the case $\frac{1}{q_{n+2}} < |p_n - q_n \mathrm{x}| < \frac{1}{q_{n+1}}$ $(n \leq$ N-2) by an equality. In this case shows that $|p_n - q_n \mathrm{x}|$ decreases steadily as n increases, Since q_n increases steadily, |x - $\frac{p_n}{q_n}$| decreases steadily.

We may sum up the most important conclusion in the following theorem

i.e. If N >1, n >0 then the differences x - $\frac{p_n}{q_n}$, q_nx - p_n = $\frac{(-1)^n \delta_n}{q_{n+1}}$, where $0 < \delta_n < 1$ ($1 \leq n \leq$ N-2), $\delta_{N-1} = 1$, |x - $\frac{p_n}{q_n}$| $\leq \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$ for n ≤ N-1 with inequality in both places except when n = N – 1.

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