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Fundamental Group of the Torus and the Dunce Cap

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ABSTRACT

In topology, the dunce cap is a compact topological space formed by taking a solid triangle and gluing all three sides together, with the orientation of one side reversed. Simply gluing two sides oriented in the same direction would yield a cone much like the layman's dunce cap, but the gluing of the third side results in identifying the base of the cap with a line joining the base to the point. For the fundamental group, use the fact that we can find a homotopy between X and the wedge of S^2 and S^1 (by moving the points where the chord joins S^2 so they coincide). The fundamental group of the wedge S^2 and S^1 is the free product of $\pi^1(S^1)$ and $\pi^1(S^2) = 1$, which is $\pi^1(S^1) = Z$. In this paper we prove that the fundamental group of the n-fold dunce cap is a cyclic group of order n and the fundamental group of the torus is a free abelian group of rank 2.

Key Words: Free Group, Deformation Retraction, Simply Connected Set, Covering Map.

Definition: Let X be a Housdorf space that is the union of the subspaces $S_1, S_2, \ldots, S_n$ each of which is homeomorphic to the unit circle s^1. Assume that there is a point p of X such that $S_i \cap S_j = \{p\}$ whenever $i \neq j$. Then X is called the wedge of the circles $S_1, S_2, \ldots, S_n$.

Note that each space S_i being compact is closed in X. Also note that X can be embedded in the plane, If C_i denote the circle of radius i in R^2 with center at (i,0) then X is homeomorphic to $C_1 \cup C_2 \cap \ldots \cup C_n$.

Theorem: Let X be the wedge of the circles $S_1, S_2, \ldots, S_n$, and p be the common point of these circles.Then π_1(X,p) is a free group. If f_i is a loop in S_i that represents a generator of $\pi_1(S_i$,p), then the loops $f_1, f_2, \ldots, f_n$ represent a system of free generators for π_1(X,p).

Proof: We prove this theorem by using induction. If n = 1 the result is obvious.

Let X be the wedge of the circles $S_1, S_2, \ldots, S_n$, with p be the common point. Chose a point q_i of S_i different from p for each i. Then the set $W_i = S_i - q_i$, and let $U = S_1 \cup W_2 \cup \ldots \cup W_n$ and $V = W_1 \cup S_2 \cup \ldots \cup S_n$. Then $U \cap V = W_1 \cup W_2 \cup \ldots \cup W_n$. We can visualize from the following figure 1.

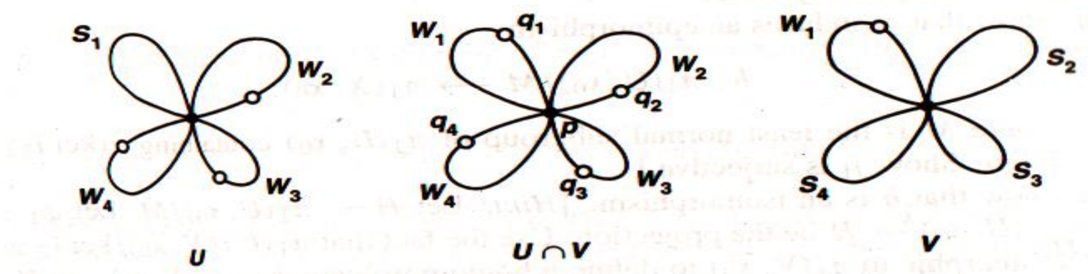

Figure 1

The space W_i is homeomorphic to an open interval, so it has the point p as a deformation retract: Let $F_i: W_i \times \mathrm{I} \to W_i$ be the deformation retraction. The maps F_i fit together to define a map F : (U∩V) × I → U∩V that is a deformation retraction of U∩V on to p.(To show that F is continuous, we note that because S_i is a closed subspace of X, the space $W_i = S_i - q_i$ is a closed subspace of U∩V, so that $W_i \times$ I is a closed subspace of (U∩V) × I.) Then by Pasting lemma it follows that U∩V is simply connected, so that $\pi_1(\mathrm{X}, \mathrm{p})$ is the free product of the groups $\pi_1(\mathrm{U}, \mathrm{p})$ and $\pi_1(\mathrm{V}, \mathrm{p})$, relative to the monomorphisms induced by inclusions.

A similar argument shows that S_1 is a deformation retract of U and $S_2 \cup \ldots \cup S_n$ is a deformation retract of V. It follows that $\pi_1(\mathrm{U}, \mathrm{p})$ is infinite cyclic, and the loop f_1 represents a generator. It also follows by using the induction hypothesis, that $\pi_1(\mathrm{V}, \mathrm{p})$ is a free group, with the loops $f_2, \ldots \ldots, f_{\mathrm{n}}$ representing a system of free generators. We generalize this result to a space X that is the union of infinitely many circles having a point in common. Here we must be careful about the topology of X.

Definition: Let X be a space that is the union of the subspaces X_α, for $\alpha \in$ J. The topology of X said to be Coherent with the subspaces X_α provided a subset C of X is closed in X if C $\cap X_\alpha$ is closed in X_α for each α. An equivalent condition is that a set be open in X if its intersection with each X_α is open in X_α.

Remark: If X is the union of finitely many closed subspaces $X_1, X_2, \ldots\ldots, X_n$, then the topology of X is automatically coherent with these subspaces, since if C $\cap X_i$ is closed in X_i, it is closed in X, and C is the finite union of the sets C $\cap X_i$.

Definition: Let X be a space that is the union of the subspaces S_α, for $\alpha \in$ J, each of which is homeomorphic to the unit circle. Assume there is a point p of X such that $S_\alpha \cap S_\beta$ = {p} whenever $\alpha \neq \beta$. If the topology of X is coherent with the subspaces S_{α}, for $\alpha \in$ J, then X is called the wedge of the circles S_{α}.

Remark: In the finite case, the definition involved the Hausdorff condition instead of the coherence condition; in that case the coherence condition followed. In the infinite case, this would no longer be true so we include the coherence condition as part of the definition. We would include the Hausdorff condition as well, but that is no longer necessary for it follows from the coherence condition.

Lemma: Let X be the wedge of the circles S_{α}, for $\alpha \in$ J. Then X is normal. Furthermore any compact subspace of X is contained in the union of finitely many circles S_{α}.

Proof: It is clear that one-point sets are closed in X. Let A and B be disjoint closed subsets of X; assume that B does not contain p. Choose disjoint subsets U_{α} and V_{α} of S_{α} that are open in S_{α} and contain {p}∪(A∩ S_{α}) and B∩ S_{α}, respectively. Let U = ∪ U_{α} and V = ∪ V_{α} then U and V are disjoint. Now U∩ S_{α}= U_{α} because all the sets U_{α} contain p and V∩ S_{α}= V_{α} because no set V_{α} contains p. Hence U and V are open in X as desired. Thus X is normal.

Now let C be a compact subspace of X. For each α for which it is possible, choose a point x_0 of C∩ ($S_{\alpha} -$ p). The set D = $\{x_{\alpha}\}$ is closed in X, because its intersection with each space S_{α} is a one point set or is empty set. For the same reason, each subset of D is closed in X. Thus D is a closed discrete subspace of X contained in C; since C is limit point compact, D must be finite.

Theorem: Let X be a wedge of the circles S_{α}, for $\alpha \in$ J; Let p be the common point of these circles. Then $\pi_1(\mathrm{X}, \mathrm{p})$ is a free group. If f_{α} is a loop in S_{α} representing a generator of $\pi_1(S_{\alpha}, \mathrm{p})$, then the loops $\{f_{\alpha}\}$ represent a system of free generators for $\pi_1(\mathrm{X}, \mathrm{p})$.

Proof: Let i_{α}: $\pi_1(S_{\alpha}, \mathrm{p}) \rightarrow \pi_1(\mathrm{X}, \mathrm{p})$ be the homeomorphism induced by inclusion; Let G_{α} be the image of i_{α}. Note that if f is any loop in X based at p, then the image set of f is compact, so that f lies in some finite union of subspaces S_{α}. Further more if f and g are two loops that are path homotopic in X, then they are actually path homotopic in some finite union of the subspaces S_{α}.

It follows that the groups $\{G_{\alpha}\}$ generate $\pi_1(\mathrm{X}, \mathrm{p})$. For if f is a loop in X, then f lies in $S_{\alpha_1} \cup S_{\alpha_2} \cup \ldots \ldots \ldots \cup S_{\alpha_n}$ for some finite set of indices; then this implies that [f] is a product of elements of the groups $G_{\alpha_1}, G_{\alpha_2}, \dots\dots\dots, G_{\alpha_n}$. Similarly it follows that i_β is a monomorphism. For if f is a loop in S_β that is path homotopic in X to a constant, then f is path homotopic to a constant in some finite union of spaces S_α. So that f is path homotopic to a constant in S_β.

Finally suppose there is a reduced non empty word w = $(g_{\alpha_1}, g_{\alpha_2}, \dots\dots\dots, g_{\alpha_n})$ in the elements of the groups G_α that represents the identity element of $\pi_1(X, p)$. Let f be a loop in X whose path homotopy class is represented by w. Then f is path homotopic to a constant in X, so that it is path homotopic to a constant in some finite union of subspaces S_α. This is a contradiction. Hence the loops $\{f_\alpha\}$ represent a system of free generators for $\pi_1(X, p)$.

Example: Let C_n be the circle of radius $\frac{1}{n}$ in R^2 with center at $(\frac{1}{n}, 0)$. Let X is the subspace of R^2 that is the union of these circles; then X is the union of a countably infinite collection of these circles, each pair of which intersects in the origin p. However X is not the wedge of the circles C_n, we call X the infinite earring.

Lemma: Given an index set J, there exists a space X, that is a wedge of circles S_α for $\alpha \epsilon$ J.

Proof: Give the set J the discrete topology, and let E be the product space $S^1 \times J$.Chose a point $b_0 \in S^1$, and let X be the quotient space obtained from E by collapsing the closed set P = $b_0 \times J$ to a point p. Let π : E → X be the quotient map, and S_α = $(S^1 \times \alpha)$. We show that each S_α is homeomorphic to S^1 and X is the wedge of the circles S_α.

Note that if C is closed in $S^1 \times \alpha$, then $\pi(C)$ is closed in X. For $\pi^{-1}\pi$ (C) = C if the point $b_0 \times \alpha$ is not in C, and $\pi^{-1}\pi$ (C) = C, if the point $b_0 \times \alpha$ is not in C, and $\pi^{-1}\pi$ (C) = C∪ P otherwise. In either case $\pi^{-1}\pi$ (C) is closed in $S^1 \times J$, so that π (C) is closed in X.

It follows that S_α is itself closed in X, since $S^1 \times \alpha$ is closed in $S^1 \times J$, and that π maps $S^1 \times \alpha$ homeomorphically onto S_α. Let π_α be this homeomophism.

To show that X has the topology coherent with the subspaces S_α, let D ⊂ X and suppose that D∩ S_α is closed in S_α for each α. Now π^{-1} (D) ∩ $(S^1 \times \alpha)$ = ${\pi_\alpha}^{-1}$(D∩ S_α); the latter set is closed in $S^1 \times \alpha$ because π_α is continuous. Then π^{-1} (D) is closed in $S^1 \times J$, so that D is closed in X by definition of the quotient topology.

Adjoining a Two-cell: If we restrict the covering map p$\times p$ to the unit square then we obtain a quotient map π: $I^2 \rightarrow$ T where T = $S^1 \times S^1$ the fundamental group of torus. But it maps BdI^2onto the subspace A = ($S^1 \times b_0$) ∪($b_0 \times S^1$), which is the wedge of two circles and maps the rest of I^2 bijectively onto T-A. Thus T can be thought of as the space obtained by pasting the edges of the square I^2 onto the space A. The process is of constructing a space by pasting the edges of a polygonal region in the plane onto another space is quite useful. We here compute the fundamental group of such a space.

Theorem: Let X be a Hausdorff space, A be a closed path connected sub-space of X. Suppose that there is a continuous map h: $B^2 \rightarrow$ X that maps IntB^2 bijectively onto X-A and maps S^1 = BdB^2 into A. Let p ∈ S^1 and let a = h(p); let k: ($S^1 \times$ p) → (A,a) be the map obtained by restricting h. Then the homeomorphism i_*: π_1(A,a) $\rightarrow \pi_1$(X,a) induced by inclusion is surjective and its kernel is the least normal subgroup of π_1(A,a) containing the image of k$_*$: $\pi_1(S^1$,p) → π_1(A,a).

Proof: Step I. Let the origin O is the center point of B^2, Let x_0 be the point h(0) of X. If U is the open set then U = X -x_0 of X, we show that A is a deformation retract of U.

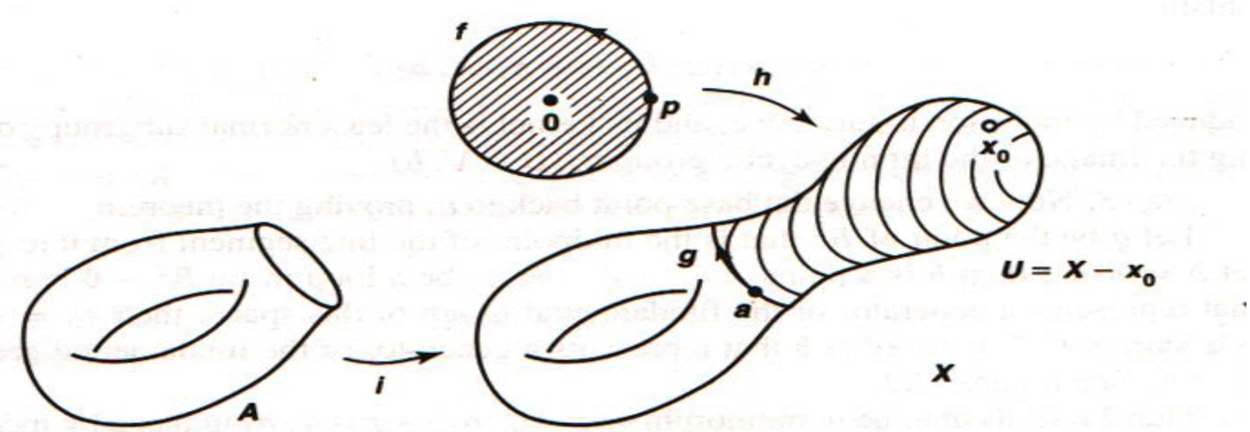

Figure 2

Let C = h (B^2), and let π: B^2 → C be the map obtained by restricting the range of h. Consider the map $\pi \times id$: $B^2 \times I$ → C$\times I$, it is a closed map because $B^2 \times I$ is compact and C$\times I$ is Hausdorff space, therefore it is a quotient map. Its restriction π': $(B^2 - 0) \times I \rightarrow$ (C$-x_0) \times I$ is also a quotient map; Since its domain is open in $B^2 \times I$ and is saturated with

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respect to $\pi \times id$. There is a deformation retraction of $B^2 - 0$ onto S^1, it induces, via the quotient map π', a deformation retraction of C-x_0 onto $\pi(S^1)$. We extend this deformation retraction to all of U$\times I$ by letting it keep each point of A fixed during the deformation. Thus A is a deformation retraction of U.

It follows that the inclusion of A in to U induces an isomorphism of fundamental groups. Our theorem then reduces to the following statement:

Let f be a loop whose class generates $\pi_1(S^1, \text{p})$. Then the inclusion of U into X induces an epimorphism $\pi_1(\text{U}, \text{a}) \to \pi_1(\text{X}, \text{a})$ whose kernel is the least normal subgroup containing the class of the loop g = hof.

Step 2. In order to prove this result, it is convenient to consider first the homomorphism $\pi_1(\text{U}, \text{b}) \to \pi_1(\text{X}, \text{b})$ induced by inclusion relative to a base point b that does not belong to A. Let b be any point of U- A. Write X as the union of the open sets U and V = X – A = $\pi(IntB^2)$. Now U is the path connected, since it has A as a deformation retracts. Because π is a quotient map, its restriction to IntB^2 is also a quotient map and hence a homeomorphism, thus V is simply connected. The set $\text{U} \cap \text{V} = \text{V} - x_0$ is homeomorphic to Int$B^2 - 0$, so it is path connected and its fundamental group is infinite cyclic. Since b is a point of U ∩V, it implies that the homomorphism $\pi_1(\text{U}, \text{b}) \to \pi_1(\text{X}, \text{b})$ induced by inclusion is surjective, and its kernel is the least normal subgroup containing the image of the infinite cyclic group $\pi_1(\text{U} \cap \text{V}, \text{b})$.

Step 3. Now we change the base point back to a and proving the theorem.

Let q be the point of B^2 that is the midpoint of the line segment from 0 to p, and let b = h(q), then b is a point of U ∩V. Let f_0 be a loop in Int$B^2 - 0$ based at q that represents a generator of the fundamental group of this space, then g_0 = hof_0 is a loop in U ∩V based at b that represents a generator of the fundamental group of U ∩V.

Step 2 tells us that the homomorphism $\pi_1(\text{U}, \text{b}) \to \pi_1(\text{X}, \text{b})$ induced by inclusion is surjective and its kernel is the least normal subgroup containing the class of the loop g_0 = hof_0. To obtain the analogous result with base point a we proceed as follows.

Let γ be the straight-line path in B^2 from q to p, let δ be the path $\delta = ho\gamma$ in U from b to a. The isomorphism induced by the path δ commute with the homomorphisms induced by inclusion in the following diagram

$$\pi_1(\mathrm{U,b}) \rightarrow \pi_1(\mathrm{X,b})$$
$$\downarrow \delta \qquad \downarrow \delta$$
$$\pi_1(\mathrm{U,a}) \rightarrow \pi_1(\mathrm{X,a})$$

Therefore the homomorphism of $\pi_1(\mathrm{U,a})\, into\, \pi_1(\mathrm{X,a})$ induced by the inclusion is surjective, and its kernel is the least normal subgroup containing the element δ. The loop f_0 represents a generator of the fundamental group of IntB – 0 based at q. Then the loop $\gamma * (f_0 * \gamma)$ represents a generator of the fundamental group of $B^2 - 0$ based at p. Therefore, it is path homotopic either to f or its reverse; suppose the former. Following this path homotopy by the map h, we see that $\delta * (g_0 * \delta)$ is path homotopic in U to g. Hence the proof.

Fundamental group of the Torus and the Dunce Cap:

Now we have to apply the results of the above given section to compute the fundamental groups of Torus and the Dunce Cap.

Theorem: The fundamental group of the torus has a presentation consisting of two generators α, β and a single relation $\alpha\beta\alpha^{-1}\beta^{-1}$.

Proof: Let X = $S^1 \times S^1$ be the torus, and let h: $I^2 \rightarrow$ X be obtained by restricting the standard covering map p×p: R×R → $S^1 \times S^1$. Let p be the point (0,0) of BdI^2,let a = h(p) and let
A = h(BdI^2). Then the above said theorem is satisfied.

The space A is the wedge of two circles, so that the fundamental group of A is free. Indeed if we let a_0 be the path a_0(t) = (t,0) and $b_0$be the path b_0(t) = (0,t) in BdI^2, then the paths α =hob_0 and β = hob_0 are loops in A such that $[\alpha]$ and $[\beta]$ form a system of free generators for $\pi_1(\mathrm{A,a})$. See Fig 3

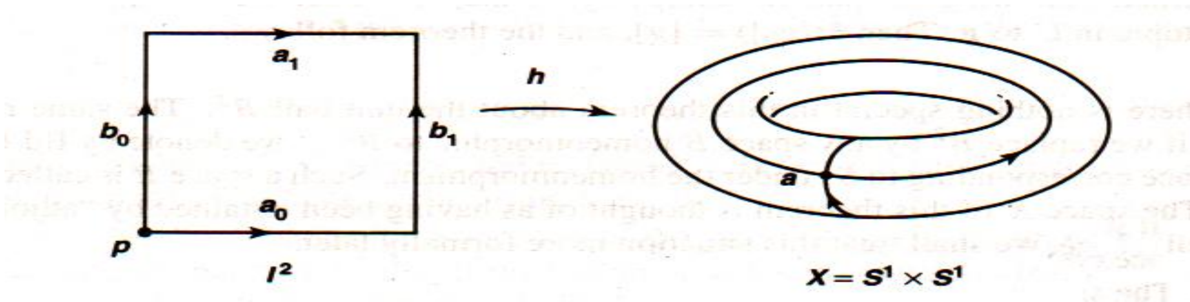

Figure 3

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Now let a_1 and b_1 be the paths a_1(t) = (t,1) and b_1(t) = (1,t) in BdI^2. Consider the loop f in BdI^2 defined by the equation f = a_0*(b_1*($\overline{a_1}$*$\overline{b_0}$)). Then f represents a generator of π_1(BdI^2, p); and the loop g = hof equals the product α*(β*($\overline{\alpha}$*$\overline{\beta}$)). Hence from the above theorem π_1(X, a) is the quotient of the free group on the free generators [α] and [β] by the least normal subgroup containing the element [α] [β] $[\alpha]^{-1}$ $[\beta]^{-1}$.

Remark: The fundamental group of the torus is a free abelian group of rank 2.

Let G be a free group on generators α, β and let N be the least normal subgroup containing the element $\alpha\beta\alpha^{-1}\beta^{-1}$. Because this element is a commutator, N is contained in the commutator subgroup [G,G] of G. On the other hand, G/N is abelian; for it is generated by the coset αN and βN, and these elements of G/N commute. Therefore N contains the commutator subgroup of G. Hence G/N is a free abelian group of rank 2.

Definition: Let n be a positive integer with n > 1. Let r: $S^1 \to S^1$ be a rotation through the angle $2\pi/n$, mapping the point ($\cos\theta, \sin\theta$) to the point (cos(θ+$2\pi/n$), sin ($\theta + 2\pi/n$)). Form a quotient space X from the unit ball B^2 by identifying each point x of S^1 with the points r(x), r^2(x),….., r^{n-1}(x). We shall show that X is a compact Hausdorff space; we call it the n-fold dunce cap.

Let π: $B^2 \to X$ be the quotient map; we show that π is a closed map. In order to do this we must show that if C is a closed set of B^2, then $\pi^{-1}\pi(C)$ is also closed in B^2; it then follow from the definition of the quotient topology that $\pi(C)$ is closed in X. Let C_0 = C∩ S^1; it is closed in B^2. The set $\pi^{-1}\pi(C)$ equals the union of C and the sets $r(C_0)$, $r^2(C_0)$,….., $r^{n-1}(C_0)$, all of which are closed in B^2, because r is a homomorphism. Hence $\pi^{-1}\pi(C)$ is closed in B^2, as desired.

Lemma: Let π: E → X be a closed quotient map. If E is normal, then so is X.

Proof: Assume that E is normal. One point sets are closed in X because one-point sets are closed in E. Now let A and B be disjoint closed sets of X. Then $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are disjoint closed sets of E. Chose disjoint open sets U and V of E containing $\pi^{-1}(A)$ and $\pi^{-1}(B)$ respectively. It is tempting to assume that π(U) and $\pi(V)$ are open sets about A and B that we are seeking. But they are not. For they need not be open (π is not necessarily an open map, and they need not be disjoint.

So we proceed as follows. Let C = E-U and let D = E – V. Because C and D are closed sets of E, the sets $\pi(C)$ and $\pi(D)$ are closed in X. Because C contains no point of π^{-1}(A), the set $\pi(C)$ is disjoint from A. Then U_0 = X - $\pi(C)$ is an open set of X containing A. Similarly V_0 = X - $\pi(D)$ is an open set of X containing B. Further more U_0 and V_0 are disjoint.
For if x ∈ U_0 then π^{-1}(x) is disjoint from C, so that it is contained in U. Similarly if x ∈ V_0 then π^{-1}(x) is disjoint from C, so that it is contained in V. Since U and V are disjoint so that U_0 and V_0.

Remark: Here note that the 2-fold dune cap is a space we have seen before, it is hoeomorphic to the projective plane P^2. To verify this fact recall that $P^2$was defined to be the quotient space obtained from S^2 by identifying x with –x for each x. Let p: $S^2 \rightarrow P^2$ be the quotient map. If we take standard homeomophism i of B^2 with the upper hemisphere of S^2, given by the equation i(x,y) = (x, y, $(1 - x^2 - y^2)^{\frac{1}{2}}$) and follow it by the map p. Hence we get a map π: $B^2 \rightarrow P^2$ that is continuous, closed and surjective. On intB it is injective and for each x∈ S^1, it maps x and –x to the same point. Hence it induces a homeomorphism of the n-fold dunce cap is just what you might expect from our computation for P^2.

Theorem: The fundamental group of the n-fold dunce cap is a cyclic group of order n.

Proof: Let h: $B^2 \rightarrow X$ be the quotient map, where X is the n-fold dunce cap. Set A = h(S^1).
Let p = (1,0) ∈ S^1 and a = h(p). Then h maps the arc C of S^1 running from p to r(p) onto A; it identifies the end points of C but is otherwise injective. Therefore A is homeomorphic to a circle, so its fundamental group is infinite cyclic. Indeed if γ is the path
γ(t) = (cos($2\pi t/n$), sin($2\pi t/n$) in $S^1$from p to r(p), then α = hoγ represents a generator of π_1(A,a).See Fig 4

Figure 4

Now the class of the loop

f = $\gamma * ((r o \gamma) * ((r^2 \text{o}\gamma) * \ldots \ldots * (r^{n-1} \text{o}\gamma)))$ generates $\pi_1(S^1, p)$. Since $h(r^m(x)) = h(x)$ for all x and m, the loop hof equals the n-fold product $\alpha * (\alpha * (\ldots \ldots * \alpha))$. Hence the proof.

REFERENCES

[1] Hatcher, Allen. Algebraic Topology. Cambridge of Chicago Press. 2002

[2] Livingston, Charles (1993). Knot theory. Mathematical Association of America. ISBN 0-88385-027-3

[3] Murasugi, Kunio (1996). Knot theory and its applications. Birkhäuser. ISBN 3-7643-3817-2

[4] Kawauchi, Akio (1996). A survey of knot theory. Birkhäuser. ISBN 3-7643-5124-1

[5] Lickorish, W. B. R. (1997). An introduction to knot theory. Springer. ISBN 0-387-98254-X

[6] Dehornoy, P. et al. (2000). Why are braids orderable? http://www.math.unicaen.fr/~dehornoy/Books/Why/Dgr.pdf

[7] Birman, J. S., and Brendle, T. E. Braids: a survey. In: Menasco, W., and Thistlethwaite, M. (Eds.) (2005). Handbook of knot theory. Elsevier. ISBN 0-444-51452-X.