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Fourth-order Steffensen type methods for nonlinear equations

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Abstract

In this paper, we introduce a new one-parameter family of fourth-order derivative-free iterative methods for solving nonlinear equations. The proposed family is based on central difference technique to approximate the derivative of a given function. Convergence theorem is also established to show its fourth-order convergence. Several numerical examples are included to demonstrate the efficiency of the proposed method and the results obtained are compared existing methods.

Keywords: Steffensen's method; Nonlinear equations; Iterative method; Derivative-free method.

1 Introduction

Solving nonlinear equations is one of the most important and challenging problem in science and engineering. Many problems arising in Kinetic theory of gases, elasticity, operation research and other applied areas [1–3] can be reduced to solving nonlinear equations. In this work, we mainly focusing on finding real and simple root of a nonlinear equation

$$f(x) = 0, \tag{1.1}$$

where $f : D \subset R \to R$ for an open interval D is a continuously differentiable scaler function. Generally, iterative methods are used to solve these type of problem because it is very difficult to solve them analytically. In the literature, researchers have proposed a large number of higher order iterative methods for solving nonlinear equations. (see Ostrowski [1], Traub [2] and Ortega and Rheinboldt [3]). These methods involve the evaluation of a function and its derivatives. The quadratically convergent Newton's method is the well known method for solving (1.1) can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.2}$$

In order to improve the rate of convergence, many authors have modified the Newton's method in a number of ways at the additional cost of a function evaluation and its derivatives. These methods are not applicable when the function is not known explicitly or the derivative of a function is difficult to compute. So, many authors have developed derivative-free methods which require only evaluation of a function at different points. If we replace the derivative of a function in Newton's method by forward difference approximation, i.e.,

$$f'(x_n) \simeq \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}, \tag{1.3}$$

then, Newton's method (1.2) becomes

$$x_{n+1} = x_n - \frac{(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n)}, \tag{1.4}$$

which is known as Steffensen's method [3]. Both the methods (1.2) and (1.4) are quadratically convergent and require two function evaluations but Steffensen's method is derivative free. Sharma and Goyal [4] presented two new one-parameter families of fourth-order derivative free methods using forward and backward approximations of derivatives. A faimly of Steffenden like methods was established by Jain [5] which require three function evaluations with third order of convergence. Zheng et al. [6] also considered the third-order derivative free methods which require the four function evaluations per step. By using the backward approximation of derivatives, Peng et al. [7] presented the fourth-order derivative free algorithm to solve nonlinear equations. A family of higher order derivative free methods is proposed by Cordero et al. [8] for nonsmooth equations. It is well known that central difference approximation is more efficient than forward and backward approximation. Dehghan and Hajarian [9] considered the central difference approximation to obtain the third-order derivative free methods which require four evaluations of a function. Recently, Cordero et al. [10] also established the derivative free Ostrowski method with fourth and sixth order of convergence using central difference approximation.

The aim of this paper is to propose a new one parameter family of fourth-order Steffensen type methods for solving nonlinear equations of the form (1.1). The proposed family of methods is based on central difference approximation of derivatives which requires four evaluations of a function. The convergence theorem is established to show fourth order convergence. Several numerical examples are worked out and results obtained are compared with existing methods. The fourth-order derivative-free Ostrowski method given by Cordero et al. [10] is a special case of the proposed family.

This paper is organized as follows. In section 2, we briefly describe the fourth-order derivative-free methods and its convergence analysis. In section 3, the proposed method is tested on a number of examples to demonstrate its efficiency and accuracy. Finally conclusions are included in section 4.

2 The Proposed method and its convergence analysis

In this section, we shall describe a family of fourth-order derivative-free methods which is based on the central difference approximation of derivatives. Consider the following iteration scheme of the form

$$y_n = x_n - \frac{f(x_n)}{h(x_n)}, \tag{2.1}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{h(x_n)}\frac{f(x_n)+\beta f(y_n)}{f(x_n)+\gamma f(y_n)}, \tag{2.2}$$

$$h(x_n) = \frac{f(x_n+f(x_n)) - f(x_n - f(x_n))}{2f(x_n)},$$

where $\beta, \gamma \in R$ and $h(x_n)$ is the central difference approximation to the derivative of a given function. In the following theorem, we discuss the convergence analysis of the proposed iteration scheme to establish fourth order convergence.

Theorem 2.1. *Let $x^* \in D$ be a simple zero of a sufficiently differentiable function $f : D \subseteq R \rightarrow R$ in an open interval D . If x_0 is sufficiently close the root x^*, then the iteration scheme defined by (2.2) is at least cubically convergent. However, for $\gamma = \beta - 2$, the order of convergence is four.*

Proof. Let $e_n = x_n - x^*$ be the error in nth iterate. Using Taylor series of $f(x_n)$ about x^*, we have

$$f(x_n) = [c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)], \tag{2.3}$$

where $c_j = (\frac{f^j(x^*)}{j!}), j = 1, 2, \ldots$

Now,

$$\begin{aligned} f(x_n + f(x_n)) &= (c_1 + c_1^2)\, e_n + (c_2 + 3c_1c_2 + c_1^2 c_2)e_n^2 + (2c_2^2 + 2c_1c_2^2 + c_1c_3 \\ &+ (1+c_1)^3 c_3)e_n^3 + (c_2^3 + 2c_2c_3 + 2c_1c_2c_3 + 3(1+c_1)^2 c_2c_3 + c_1c_4 \\ &+ (1+c_1)^4 c_4)e_n^4 + O(e_n^5), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} f(x_n - f(x_n)) &= (c_1 - c_1^2)\, e_n + (c_2 - 3c_1c_2 + c_1^2 c_2)e_n^2 + (-2c_2^2 + 2c_1c_2^2 \\ &+ (1-c_1)^3 c_3 - c_1c_3)e_n^3 + (c_2^3 - 2c_2c_3 - 3(1-c_1)^2 c_2c_3 \\ &+ 2c_1c_2c_3 + (1-c_1)^4 c_4 - c_1c_4)e_n^4 + O(e_n^5). \end{aligned} \tag{2.5}$$

Using equations (2.3)–(2.5), we obtain

$$\begin{aligned} \frac{f(x_n)}{h(x_n)} &= \frac{2(f(x_n))^2}{f(x_n+f(x_n)) - f(x_n - f(x_n))}, \\ &= e_n - \frac{c_2 e_n^2}{c_1} + \frac{(2c_2^2 - 2c_1c_3 - c_1^3 c_3)\, e_n^3}{c_1^2} \\ &+ \frac{(-4c_2^3 + 7c_1c_2c_3 + c_1^3 c_2c_3 - 3c_1^2 c_4 - 4c_1^4 c_4)e_n^4}{c_1^3} + O(e_n^5). \end{aligned} \tag{2.6}$$

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Therefore,

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{h(x_n)}, \\ &= x^* + \frac{c_2 e_n^2}{c_1} + \frac{(-2c_2^2 + 2c_1c_3 + c_1^3 c_3)\, e_n^3}{c_1^2} \\ &+ \frac{(4c_2^3 - 7c_1c_2c_3 - c_1^3 c_2 c_3 + 3c_1^2 c_4 + 4c_1^4 c_4) e_n^4}{c_1^3} + O(e_n^5). \end{aligned} \quad (2.7)$$

Using the Taylor's series of $f(y_n)$ about x^*, we have

$$\begin{aligned} f(y_n) &= [c_1(y_n - x^*) + c_2(y_n - x^*)^2 + c_3(y_n - x^*)^3 + c_4(y_n - x^*)^4 + O(e_n^5)], \\ &= c_2 e_n^2 + \frac{(-2c_2^2 + 2c_1c_3 + c_1^3 c_3)\, e_n^3}{c_1} + \frac{(5c_2^3 - 7c_1c_2c_3 - c_1^3 c_2 c_3 + 3c_1^2 c_4 + 4c_1^4 c_4)\, e_n^4}{c_1^2} \\ &+ O(e_n^5). \end{aligned} \quad (2.8)$$

Using the values of $f(x_n)$ and $f(y_n)$, we get

$$\begin{aligned} f(x_n) + \beta f(y_n) &= c_1 e_n + (c_2 + \beta c_2)\, e_n^2 + \left(c_3 + \beta\left(-\frac{2c_2^2}{c_1} + 2c_3 + c_1^2 c_3\right)\right) e_n^3 \\ &+ \left(c_4 + \beta\left(\frac{5c_2^3}{c_1^2} - \frac{7c_2c_3}{c_1} - c_1c_2c_3 + 3c_4 + 4c_1^2 c_4\right)\right) e_n^4 + O(e_n^5). \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} f(x_n) + \gamma f(y_n) &= c_1 e_n + (c_2 + \gamma c_2)\, e_n^2 + \left(c_3 + \gamma\left(-\frac{2c_2^2}{c_1} + 2c_3 + c_1^2 c_3\right)\right) e_n^3 \\ &+ \left(c_4 + \gamma\left(\frac{5c_2^3}{c_1^2} - \frac{7c_2c_3}{c_1} - c_1c_2c_3 + 3c_4 + 4c_1^2 c_4\right)\right) e_n^4 + O(e_n^5). \end{aligned} \quad (2.10)$$

Substituting the equations (2.6)-(2.10) in the expression (2.2), we get

$$e_{n+1} = \frac{(2c_2^2 - \beta c_2^2 + \gamma c_2^2)\, e_n^3}{c_1^2} + \frac{((1+2\beta)c_2^3 - (c_1 + c_1^3)\, c_2c_3) e_n^4}{c_1^3} + O(e_n^5). \quad (2.11)$$

Therefore, the methods defined by (2.2) are at least cubically convergent for any $\beta, \gamma \in R$. However, for $\gamma = \beta - 2$, the order of convergence of the method becomes four and the error equation is given by

$$e_{n+1} = \frac{((1+2\beta)c_2^3 - (c_1 + c_1^3)\, c_2c_3) e_n^4}{c_1^3} + O(e_n^5). \quad (2.12)$$

Thus, we have established a new one-parameter family of fourth-order derivative-free methods given as follows

$$\begin{aligned} y_n &= x_n - \frac{2(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= y_n - \frac{2f(y_n)f(x_n)}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)}. \end{aligned} \quad (2.13)$$

For $\beta = 0$, the proposed family (2.13) reduces to

$$\begin{aligned} y_n &= x_n - \frac{2(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= x_n - \frac{2(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)} \end{aligned} \tag{2.14}$$

which is a Ostrowski's fourth order derivative free method [10].

For $\beta = 1$, we have obtained a new fourth order iterative method free from derivatives which is given by

$$\begin{aligned} y_n &= x_n - \frac{2(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= y_n - \frac{2f(y_n)f(x_n)}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \frac{f(x_n) + f(y_n)}{f(x_n) - f(y_n)}. \end{aligned} \tag{2.15}$$

The above proposed method is denoted by PM. Substituting $\beta = 1$ in (2.12), its error equation is given by

$$e_{n+1} = \frac{(3c_2^3 - (c_1 + c_1^3)\, c_2 c_3) e_n^4}{c_1^3} + O(e_n^5).$$

Each member of the family (2.13) requires only four function evaluations and free from derivatives. Generally, efficiency index [11] of a method is defined as $I = p^{\frac{1}{m}}$, where p is the order of the method and m is the number of function evaluations per iteration required by the method. There are various technique available to construct the higher order iterative methods, but the complexity of the iterative scheme increases substantially. Therefore, a new index has been used to compare the various iterative methods which takes not only the number of function evaluations, but also the the number of products and quotients involved in each step. The computational efficiency index is defined as $CI = p^{1/(m+op)}$, where op denotes the number of products and quotients per iteration. Table 1 exhibits the order of convergence, efficiency index and the computational efficiency index of some derivative free methods. From the Table 1, it is clear that for the efficiency index

$$I_{JM} > I_{PM} = I_{CO} = I_{SM} > I_{ZM} = I_{DM},$$

and the computational efficiency index :

$$CI_{PM} = CI_{CO} > CI_{SM} > CI_{DM} = CI_{JM} > CI_{ZM}.$$

3 Numerical Examples

In this section, we have worked out our proposed method (PM) on the following nonlinear functions with starting iterate x_0.

Table 1 Order and efficiency indices of some derivative free methods

Method	Order	Efficiency index	Comp. efficency index
Steffensen (SM)	2	$2^{1/2}$	$2^{1/(2+2)}$
Jain (JM)	3	$3^{1/3}$	$3^{(1/(3+6))}$
Zheng et al. (ZM)	3	$3^{1/4}$	$3^{1/(4+6)}$
Dehghan-Hajarian (DM)	3	$3^{1/4}$	$3^{1/(4+4)}$
Cordero (CO)	4	$4^{1/4}$	$4^{1/(4+4)}$
Proposed method (PM)	4	$4^{1/4}$	$4^{1/(4+4)}$

$$f_1(x) = x^2 - e^x - 3x + 2,\ x^* \approx 0.257530285439861$$
$$f_2(x) = \sin^2(x) - x^2 + 1,\ x^* \approx 1.404491648215341$$
$$f_3(x) = x^3 + 4x^2 - 10,\ x^* \approx 1.36523001314096$$
$$f_4(x) = (1 + \cos(x))(e^x - 2),\ x^* \approx 0.693147180559945$$
$$f_5(x) = xe^x - 0.1,\ x^* \approx 0.111832559158962$$
$$f_6(x) = \cos(x) - x,\ x^* \approx 0.739085133215161$$
$$f_7(x) = e^{-x} + \cos(x),\ x^* \approx 1.746139530408013$$

Numerical results obtained by the proposed method (PM) for $\beta = 1$ are compared with Newton's method (NM), Steffensen's method (SM) [3], and Dehghan and Hajarian's method (DM) [9] respectively. All the numerical computations have been performed in MAPLE with double arithmetic precision and the stopping criteria used is $|x_{n+1} - x_n| < 10^{-17}$. The number of iterations (IT), the value of $f(x_n)$ and distance between two successive approximation $|x_{n+1} - x_n|$ to approximate the root x^* for NM, SM, DM and PM are shown in the Table 2 with suitable initial approximation x_0. The Table 2 exhibits that the proposed method gives the improved results in terms of computational speed and accuracy.

4 Conclusions

A new one-parameter family of fourth-order derivative-free methods has been proposed for solving nonlinear equations. Central difference approximation technique has been used for approximating the derivative of a given function. The convergence analysis of the proposed family has been discussed in a simpler way to show its fourth-order convergence. Numerical results shows that the proposed method provides the improved results in terms of computational speed and accuracy. The method developed in [10] is seen as a special case of the family.

Table 2 Comparison of various iterative methods

	IT	$f(x_n)$	$\lvert x_{n+1} - x_n \rvert$
$f_1, x_0 = 2$			
NM	6	$2.9e{-55}$	$9.1e{-28}$
SM	7	$2.5e{-43}$	$5.1e{-22}$
DM	18	$2.2e{-81}$	$3.2e{-27}$
PM	5	$3.9e{-140}$	$1.9e{-35}$
$f_2, x_0 = 1$			
NM	7	$1.0e{-50}$	$7.3e{-26}$
SM	7	$1.0e{-62}$	$6.1e{-32}$
DM	5	$1.2e{-71}$	$1.5e{-24}$
PM	4	$3.0e{-103}$	$1.9e{-26}$
$f_3, x_0 = 1.5$			
NM	5	$1.2e{-37}$	$1.2e{-19}$
SM	8	$1.0e{-35}$	$2.1e{-19}$
DM	4	$5.4e{-54}$	$8.8e{-19}$
PM	4	$1.2e{-120}$	$3.1e{-31}$
$f_4, x_0 = 2$			
NM	5	$6.0e{-46}$	$3.5e{-23}$
SM	6	$1.2e{-52}$	$7.6e{-27}$
DM	4	$1.3e{-79}$	$9.8e{-27}$
PM	4	$1.7e{-167}$	$1.8e{-42}$
$f_5, x_0 = 0.2$			
NM	6	$2.6e{-65}$	$5.6e{-33}$
SM	6	$2.2e{-49}$	$3.8e{-25}$
DM	4	$3.1e{-69}$	$1.2e{-23}$
PM	4	$3.3e{-225}$	$6.3e{-57}$
$f_6, x_0 = 1$			
NM	6	$1.5e{-41}$	$5.6e{-33}$
SM	6	$3.1e{-44}$	$3.8e{-25}$
DM	4	$8.0e{-102}$	$1.2e{-23}$
PM	4	0	$6.4e{-66}$
$f_7, x_0 = 2$			
NM	5	$2.6e{-42}$	$3.9e{-21}$
SM	5	$2.3e{-71}$	$2.8e{-35}$
DM	4	$5.1e{-90}$	$4.6e{-30}$
PM	4	$1.8e{-238}$	$7.4e{-60}$

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